MATH 307: Problem Set #3 Solutions

Due on: May 3, 2015

Problem 1 *Autonomous Equations*

Recall that an equilibrium solution of an autonomous equation is called *stable* if solutions lying on both sides of it tend toward it; is called *unstable* if solutions lying on both sides tend away from it; and is called *semistable* if solutions lying on one side of it tend toward it, while solutions on the other side tend away. In Prob 2.i-2.vi, please do each of the following

(a) Sketch a graph of $f(y)$ versus y

(b) Determine the critical (equilibrium) points

(c) Classify the critical points as stable, semistable, or unstable

(d) Draw the phase line and sketch several graphs of the solution in the ty-plane.

(i) $dy/dt = ay + by^2$, where $a > 0$, $b > 0$ and $y_0 \geq 0$

(ii) $dy/dt = ay + by^2$, where $a > 0$, $b > 0$ and $-\infty < y_0 < \infty$

(iii) $dy/dt = e^y - 1$, $-\infty < y_0 < \infty$

(iv) $dy/dt = e^{-y} - 1$, $-\infty < y_0 < \infty$

(v) $dy/dt = y(1 - y^2)$, $-\infty < y_0 < \infty$

(vi) $dy/dt = y^2(4 - y^2)$, $-\infty < y_0 < \infty$

. . . .

Solution 1.

(i) The plots for problem 1.i are included in Figure (1). There is one unstable phase point at 0

(ii) The plots for problem 1.ii are included in Figure (2). There is one unstable phase point at 0 and one stable phase point at -1

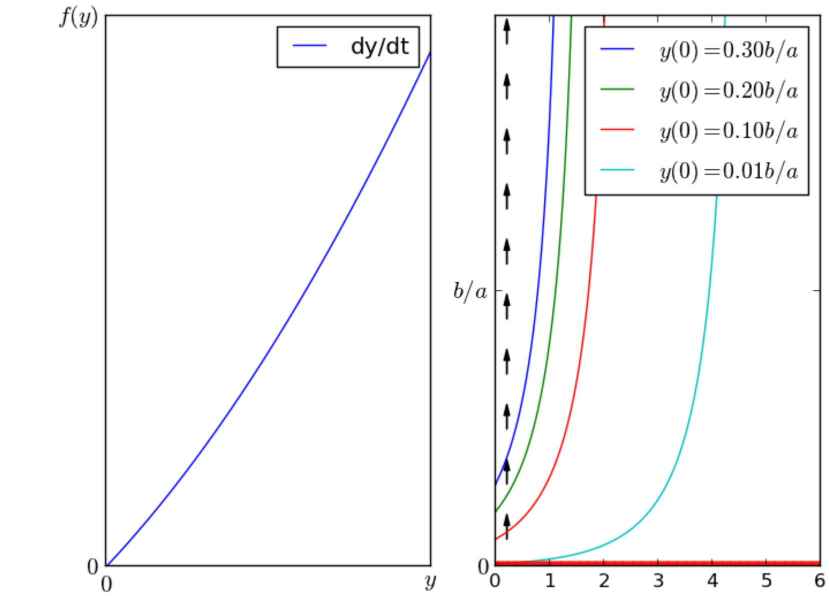

Figure 1: Plots for Problem 1.i

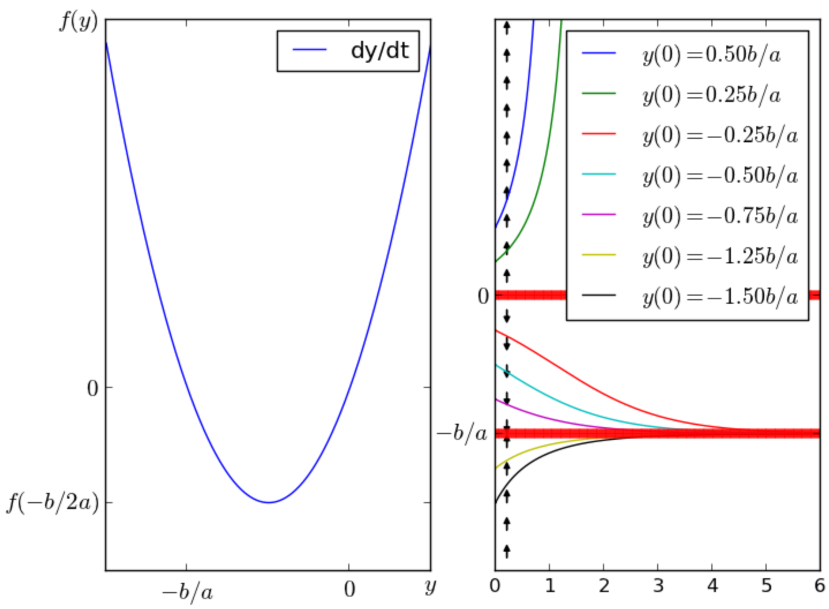

Figure 2: Plots for Problem 1.ii

(iii) The plots for problem 1.iii are included in Figure (3). There is one unstable phase point at 0

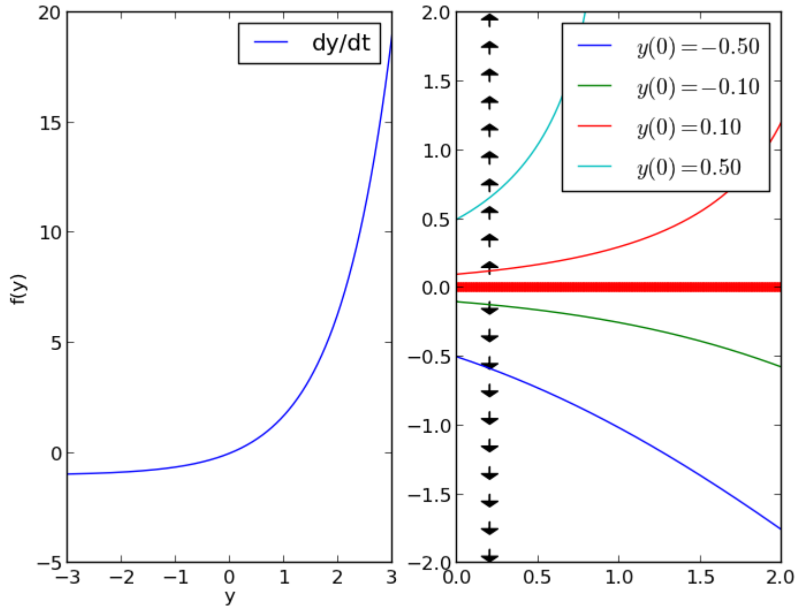

Figure 3: Plots for Problem 1.iii

(iv) The plots for problem 1.iv are included in Figure (4). There is one stable phase point at 0

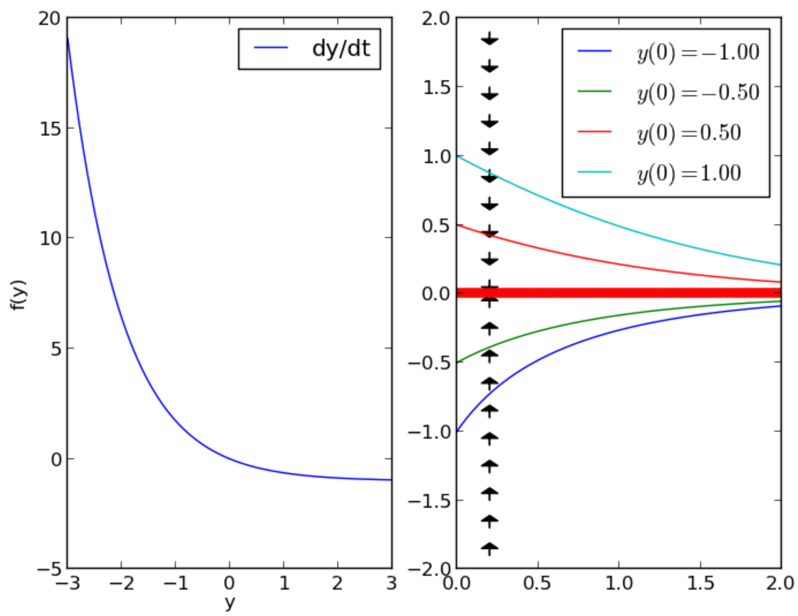

Figure 4: Plots for Problem 1.iv

(v) The plots for problem 1.v are included in Figure (5). There is one unstable phase point at 0 and two stable phase points at 1 and -1.

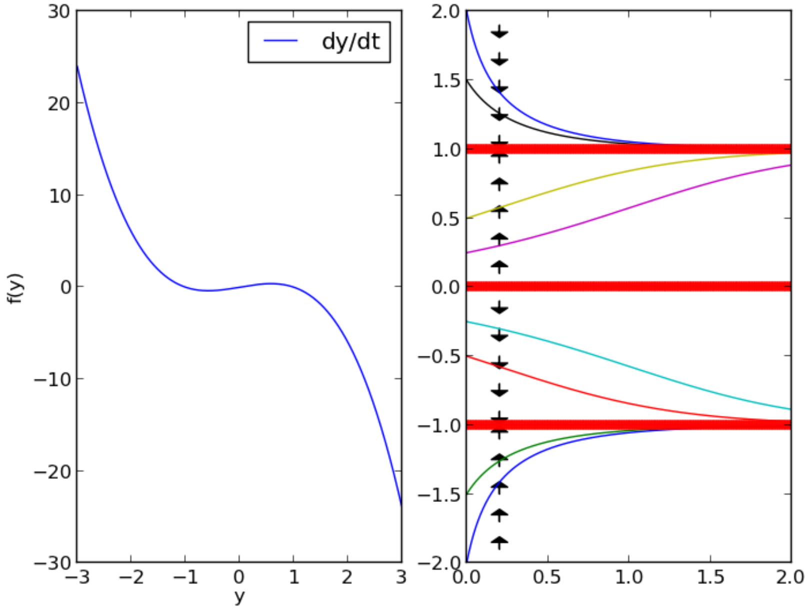

Figure 5: Plots for Problem 1.v (legend omitted for visibility)

(vi) The plots for problem 1.vi are included in Figure (6). There is one semistable phase point at 0 and one stable phase points at 2 and one unstable phase point at -2.

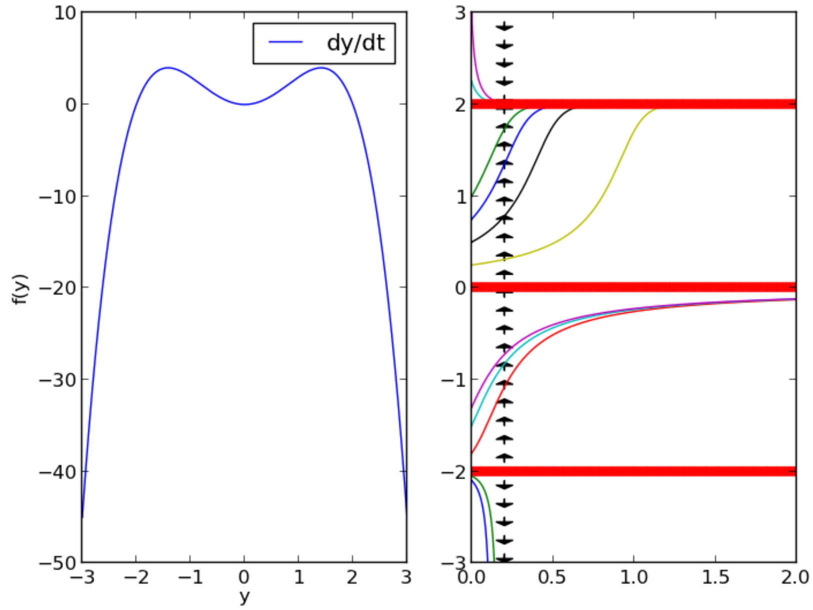

Figure 6: Plots for Problem 1.vi (legend omitted for visibility)

Problem 2 *Euler's Method*

For Prob 1.i and 1.ii please do each of the following

(a) Find approximate values of the solution of the given value problem in the interval $[0, 0.5]$ with $\Delta t = 0.100$ using Euler's method. Record your results as a table of values in your writeup.

(b) Find approximate values of the solution of the given value problem in the interval $[0, 0.5]$ with $\Delta t = 0.050$ using Euler's method. Record your results as a table of values in your writeup.

(c) Find approximate values of the solution of the given value problem in the interval $[0, 0.5]$ with $\Delta t = 0.025$ using Euler's method. Record your results as a table of values in your writeup.

(d) Find the exact solution to the initial value problem.

(e) Compare your results in parts (a), (b), (c), and (d) by plotting them all in the same graph. Be sure that your plot is clear enough that one can tell which curves correspond to each part.

(i) $y' = 2y - 1$, $y(0) = 1$

(ii) $y' = 3\cos(t) - 2y$, $y(0) = 0$

. . . .

Solution 2.

(i) The plots for problem 2.i are included in Figure (8) below. When you perform the calculation, the specific data values you should get are

t	f(t)
0.000	1.000
0.100	1.100
0.200	1.220
0.300	1.364
0.400	1.537
0.500	1.744

t	f(t)
0.000	1.000
0.050	1.050
0.100	1.105
0.150	1.165
0.200	1.232
0.250	1.305
0.300	1.386
0.350	1.474
0.400	1.572
0.450	1.679
0.500	1.797

t	f(t)
0.000	1.000
0.025	1.025
0.050	1.051
0.075	1.079
0.100	1.108
0.125	1.138
0.150	1.170
0.175	1.204
0.200	1.239
0.225	1.276
0.250	1.314
0.275	1.355
0.300	1.398
0.325	1.443
0.350	1.490
0.375	1.539
0.400	1.591
0.425	1.646
0.450	1.703
0.475	1.763
0.500	1.827

and the exact solution that you should get is

$$y = \frac{1}{2}(1 + e^{2t})$$

(ii) The plots for problem 2.i are included in Figure (8) below. When you perform the calculation, the specific data values you should get are

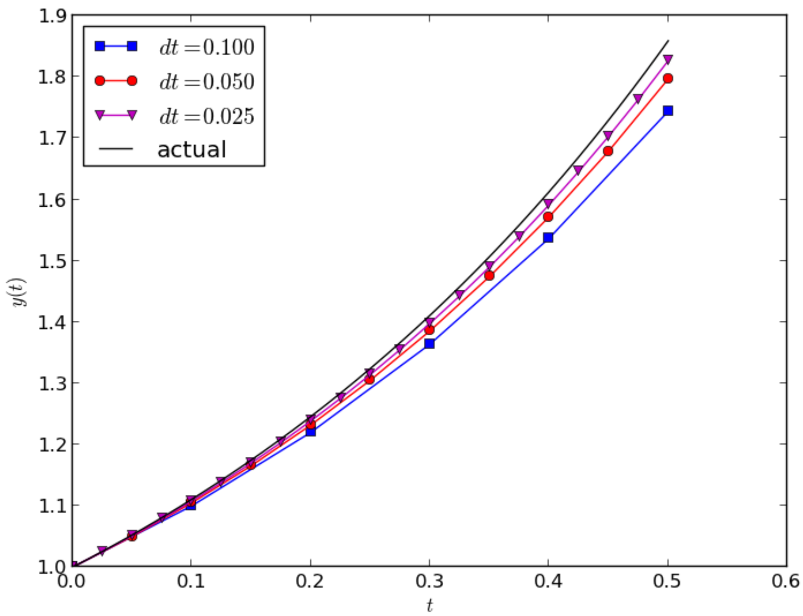

Figure 7: Plot for Problem 2.i. Notice that solutions approach actual solution as dt decreases.

t	f(t)
0.000	0.000
0.100	0.300
0.200	0.539
0.300	0.725
0.400	0.866
0.500	0.969

t	f(t)
0.000	0.000
0.050	0.150
0.100	0.285
0.150	0.406
0.200	0.513
0.250	0.609
0.300	0.693
0.350	0.767
0.400	0.832
0.450	0.887
0.500	0.933

t	f(t)
0.000	0.000
0.025	0.075
0.050	0.146
0.075	0.214
0.100	0.278
0.125	0.339
0.150	0.396
0.175	0.450
0.200	0.502
0.225	0.550
0.250	0.596
0.275	0.639
0.300	0.679
0.325	0.717
0.350	0.752
0.375	0.785
0.400	0.815
0.425	0.844
0.450	0.870
0.475	0.894
0.500	0.916

and the exact solution that you should get is

$$y = \frac{3}{5}\sin(t) + \frac{6}{5}\cos(t) - \frac{6}{5}(e^{-2t})$$

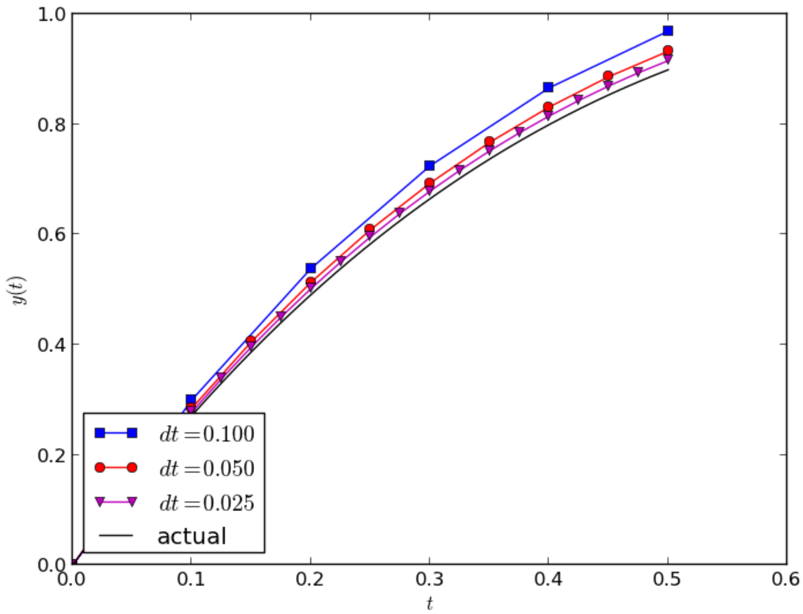

Figure 8: Plot for Problem 2.ii. Notice that solutions approach actual solution as dt decreases.

Problem 3 *Exact Equations*

In each of the following, determine if the equation is exact. If it is exact, then find the solution.

(i) $(2x + 4y) + (2x - 2y)y' = 0$

(ii) $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$

(iii) $\frac{dy}{dx} = -\frac{ax-by}{bx-cy}$

(iv) $(e^x \sin y + 3y)dx - (3x - e^x \sin y)dy = 0$

(v) $(y/x + 6x)dx + (\ln(x) - 2)dy = 0$

(vi) $\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}} = 0$

. . . .

Solution 3.

(i) $M_y = 4$ but $N_x = 2$, so this is not exact

(ii) $M_y = 4xy + 2$ and $N_x = 4xy + 2$, so this is exact. Therefore there exists a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Thus

$$\psi(x, y) = \int (2xy^2 + 2y)\partial x = x^2y^2 + 2xy + h(y)$$

for some unknown function h. Then

$$\psi_y = \frac{\partial}{\partial y}(x^2y^2 + 2xy + h(y)) = 2x^2y + 2x + h'(y),$$

and since $\psi_y = N$, we must have

$$2x^2y + 2x + h'(y) = 2x^2y + 2x.$$

Hence $h'(y) = 0$, meaning that h is a constant which we can take to be zero. Thus $\psi(x, y) = x^2y^2 + 2xy$, and the solution to to the original differential equation is $\psi(x, y) = C$, ie.

$$x^2y^2 + 2xy = C$$

where C is an arbitrary constant.

(iii) We rewrite this equation as

$$\frac{ax - by}{bx - cy} + y' = 0$$

Then $M_y = \frac{(ac-b^2)x}{(bx-cy)^2}$ and $N_x = 0$, so this is not exact

(iv) $M_y = e^x \cos(y) + 3$ and $N_x = -3 + e^x \sin(y)$, so this is not exact

(v) $M_y = 1/x$ and $N_x = 1/x$, so this is exact Therefore there exists a function $\psi(x,y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Thus

$$\psi(x,y) = \int (\ln(x) - 2)\partial y = y\ln(x) - 2y + g(x)$$

for some unknown function g. Then

$$\psi_x = \frac{\partial}{\partial x}(y\ln(x) - 2y + g(x)) = y/x + g'(x),$$

and since $\psi_x = M$, we must have

$$y/x + g'(x) = y/x + 6x$$

Hence $g'(x) = 6x$, so we can take $g(x) = 3x^2$. Thus $\psi(x,y) = y\ln(x) - 2y + 3x^2$, and the solution to to the original differential equation is $\psi(x,y) = C$, ie.

$$y\ln(x) - 2y + 3x^2 = C$$

where C is an arbitrary constant.

(vi) $M_y = \frac{-3xy}{(x^2+y^2)^{5/2}} = N_x$, so this is exact. Therefore there exists a function $\psi(x,y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Thus

$$\psi(x,y) = \int \left(\frac{x}{(x^2+y^2)^{3/2}}\right)\partial x = \frac{-1}{(x^2+y^2)^{1/2}} + h(y)$$

for some unknown function h. Then

$$\psi_y = \frac{\partial}{\partial y}\left(\frac{-1}{(x^2+y^2)^{1/2}} + h(y)\right) = \frac{y}{(x^2+y^2)^{3/2}} + h'(y),$$

and since $\psi_y = N$, we must have

$$\frac{y}{(x^2+y^2)^{3/2}} + h'(y) = \frac{y}{(x^2+y^2)^{3/2}}$$

Hence $h'(y) = 0$, so h is constant, and we can take it to be zero. Thus $\psi(x,y) = \frac{-1}{(x^2+y^2)^{1/2}}$, and the solution to to the original differential equation is $\psi(x,y) = C$, ie.

$$\frac{-1}{(x^2+y^2)^{1/2}} = C$$

where C is an arbitrary constant.

Problem 4 *Integrating Factors*

For each of the following, find an integrating factor and solve the given equation

(i) $y' = e^{2x} + y - 1$

(ii) $ydx + (2xy - e^{-2y})dy = 0$

(iii) $\left(3y + \frac{\sin(y)}{x^2 y}\right) dx + \left(2x + \frac{\cos(y)}{xy}\right) dy = 0$ [Hint: Try $\mu(x, y) = x^a y^b$]

(iv) $\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3\frac{y}{x}\right) \frac{dy}{dx} = 0$ [Hint: Try $\mu(x, y) = \mu(xy)$]

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Solution 4 (Partial Solution)**.**

(i) We try an integrating factor of the form $\mu = \mu(x)$. Then

$$\mu(x)y' = \mu(x)e^{2x} + \mu(x)y - \mu(x)$$

must be exact. Rewriting this, we see

$$\overbrace{-\mu(x)e^{2x} - \mu(x)y + \mu(x)}^{M(x,y)} + \overbrace{\mu(x)}^{N(x,y)} y' = 0.$$

We calculate $M_y = -\mu(x)$ and $N_x = \mu'(x)$, and since $M_y = N_x$, we must have $\mu'(x) = -\mu(x)$, ie. $\mu(x) = e^{-x}$. Hence the equation

$$-e^x - e^{-x}y + e^{-x} + e^{-x}y' = 0$$

is exact.

Thus there exists a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. This means

$$\psi(x, y) = \int e^{-x} \partial y = ye^{-x} + g(x)$$

for some unknown function g. Then

$$\psi_x = \frac{\partial}{\partial x} \left(ye^{-x} + g(x)\right) = -ye^{-x} + g'(x),$$

and since $\psi_x = M$, we must have

$$-ye^{-x} + g'(x) = -e^x - e^{-x}y + e^{-x}$$

Hence $g'(x) = -e^x + e^{-x}$, so g can be taken to be $g = -e^x - e^{-x}$. Thus $\psi(x, y) = ye^{-x} - e^x - e^{-x}$, and the solution to to the original differential equation is $\psi(x, y) = C$, ie.

$$ye^{-x} - e^x - e^{-x} = C.$$

where C is an arbitrary constant. For simplicity, we may rewrite this as

$$y = e^{2x} + 1 + Ce^x.$$

(ii) We try an integrating factor of the form $\mu = \mu(y)$. Then

$$\overbrace{\mu(y)y}^{M(x,y)}\,dx + \overbrace{(2x\mu(y)y - \mu(y)e^{-2y})}^{N(x,y)}\,dy = 0$$

must be exact. Therefore $M_y = N_x$, which means that

$$\mu'(y)y + \mu(y) = 2\mu(y)y.$$

Which simplifies to

$$\mu'(y)y = (2y-1)\mu(y),$$

a separable equation. Solving this equation, we obtain $\mu(y) = e^{2y}/y$. Therefore

$$e^{2y}dx + (2xe^{2y} - 1/y)dy$$

is exact.

Thus there exists a function $\psi(x,y)$ satisfying $\psi_x = M$ and $\psi_y = N$. This means

$$\psi(x,y) = \int e^{2y}\partial x = xe^{2y} + h(y)$$

for some unknown function h. Then

$$\psi_y = \frac{\partial}{\partial y}\left(xe^{2y} + h(y)\right) = 2xe^{2y} + h'(y),$$

and since $\psi_x = M$, we must have

$$2xe^{2y} + h'(y) = 2xe^{2y} - 1/y$$

Hence $h'(y) = -1/y$, so h can be taken to be $h = -\ln|y|$. Thus $\psi(x,y) = xe^{2y} - \ln|y|$, and the solution to to the original differential equation is $\psi(x,y) = C$, ie.

$$xe^{2y} - \ln|y| = C.$$

(iii) We try an integrating factor of the form $\mu(x,y) = x^a y^b$ for some unknown constants a, b. This means that the equation

$$\overbrace{\left(3x^a y^{b+1} + x^{a-2}y^{b-1}\sin(y)\right)}^{M(x,y)}\,dx + \overbrace{\left(2x^{a+1}y^b + x^{a-1}y^{b-1}\cos(y)\right)}^{N(x,y)}\,dy = 0$$

must be exact. We calculate

$$M_y = 3(b+1)x^a y^b + (b-1)x^{a-2}y^{b-2}\sin(y) + x^{a-2}y^{b-1}\cos(y)$$

and also

$$N_x = 2(a+1)x^a y^b + (a-1)x^{a-2}y^{b-1}\cos(y)$$

Since $M_y = N_x$, we can write $M_y - N_x = 0$, which after some algebra becomes

$$(3b - 2a + 1)x^a y^b + (b-1)x^{a-2}y^{b-2}\sin(y) + (2-a)x^{a-2}y^{b-1}\cos(y) = 0.$$

If this is true for all values of x and y, then the coefficients must all be zero, meaning

$$\begin{aligned} 3b - 2a + 1 &= 0 \\ b - 1 &= 0 \\ 2 - a &= 0 \end{aligned}$$

The solution of this system of equations is $a = 2$ and $b = 1$. Hence the equation

$$\overbrace{\left(3x^2y^2 + \sin(y)\right)}^{M(x,y)} dx + \overbrace{\left(2x^3y + x\cos(y)\right)}^{N(x,y)} dy = 0$$

is exact.

Thus there exists a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. This means

$$\psi(x,y) = \int 3x^2y^2 + \sin(y)\partial x = x^3y^2 + x\sin(y) + h(y)$$

for some unknown function h. Then

$$\psi_y = \frac{\partial}{\partial y}\left(x^3y^2 + x\sin(y) + h(y)\right) = 2x^3y + x\cos(y) + h'(y),$$

and since $\psi_x = M$, we must have

$$2x^3y + x\cos(y) + h'(y) = 2x^3y + x\cos(y)$$

Hence $h'(y) = 0$, so h is a constant, which we can take to be 0. Thus $\psi(x, y) = x^3y^2 + x\sin(y)$, and the solution to to the original differential equation is $\psi(x, y) = C$, ie.

$$x^3y^2 + x\sin(y) = C.$$

(iv) Here's how to handle this problem. Suppose that the integrating factor is of the form $\mu(xy)$. Our original equation is

$$F(x, y) + G(x, y)y' = 0$$

where $F(x, y) = 3x + 6/y$ and $G(x, y) = x^2/y + 3y/x$. Therefore the equation

$$\overbrace{\mu(xy)F(x,y)}^{M(x,y)} + \overbrace{\mu(xy)G(x,y)}^{N(x,y)}\, y' = 0$$

must be exact! This means that $\partial M/\partial y = \partial N/\partial x$. Therefore

$$x\mu'(xy)F(x,y) + \mu(xy)\frac{\partial F(x,y)}{\partial y} = y\mu'(xy)G(x,y) + \mu(xy)\frac{\partial G(x,y)}{\partial x}$$

Now gathering all the terms that have been hit with a μ on one side, and the terms that have been hit with a μ' on the other, we find

$$(xF(x,y) - yG(x,y))\mu'(xy) = \mu(xy)\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right).$$

Therefore:

$$\frac{1}{\mu(xy)}\mu'(xy) = \frac{\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}}{xF(x,y) - yG(x,y)}.$$

Now if we work it out for our particular values of F and G, and do a bit of algebra, we find that the right hand side of this equation becomes simply $\frac{1}{xy}$. Therefore

$$\frac{1}{\mu(xy)}\mu'(xy) = \frac{1}{xy}.$$

Now replacing xy with a new variable z, we gather that

$$\frac{1}{\mu(z)}\mu'(z) = \frac{1}{z}.$$

A solution to this equation is $\mu(z) = z$. Thus $\mu(xy) = xy$ is our integrating factor. Multiplying everything by this, we get

$$(3x^2y + 6x) + (x^3 + 3y^2)y' = 0$$

which is now exact. We solve it in the usual way, obtaining the family of solutions

$$x^3y + 3x^2 + y^3 = C.$$