Week 1 in Review

Intro. to Differential Equations

April 5, 2015

TOPICS

- What is a differential equation
- What is a solution to a differential equation
- First order separable equations
- First order homogeneous equations
- First order linear equations

1 What is a differential equation

Definition 1. A differential equation is an equation expressing a relationship between a function and its derivatives

Some examples of differential equations:

Example 1 (Korteg-de Vries)**.** The partial differential equation

$$u_t + u_{xxx} + 6uu_x = 0$$

can be used to describe the motion of a one-dimensional shallow water fluid flow. It's is especially interesting from a mathematical point of view, because its solutions can be precisely determined.

Example 2 (One-Dimensional Stationary Schrödinger Equation)**.** The ordinary differential equation

$$-\frac{\hbar}{2\mu}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

with $\hbar, \mu, E$ constants describes the probability distribution of a single particle in space with potential energy $V(x)$.

Example 3 (Newton's Law of Cooling)**.** The differential equation

$$\frac{dT}{dt} = -k(T - T_a)$$

describes how quickly the temperature T of your coffee will cool to the relative to the ambient temperature T_a of the room where you are drinking it.

2 What is a solution to a differential equation

Definition 2. A solution to a differential equation is a function that satisfies the differential equation.

For example, the function $y(x) = \sin(x)$ satisfies

$$y' = \cos(x)$$
$$y'' = -\sin(x) = -y$$

Therefore $\sin(x)$ is a solution to the differential equation

$$y'' + y = 0.$$

2.1 Slope Fields

For first-order differential equations, we also have a geometric interpretation of solutions in terms of slope fields.

Proposition 1. *A solution to a first order differential equation*

$$y' = f(x, y)$$

is a curve in the x, y-plane satisfying that at every point on the curve the slope of the tangent line agrees with the value of the slope field.

In other words, by thinking of the slope field as current in the sea, and we throw a bucket of dye in the ocean at a point (a,b) (and ignore the legal and moral ramifications of this) the dye will trace out a path that forms a solution to the initial value problem

$$y' = f(x, y), \quad y(a) = b.$$

3 First Order Separable Equations

First order separable equations are easy to solve – up to an integral that can be quite hard – and the process is always the same!

Definition 3. A first order separable ODE is a differential equation of the form

$$y' = f(x)g(y).$$

Example 4. (a) $xy' + y^2 \cos(x)$ is separable

(b) $y' = (x^2 - y^2)/(x + y)$ is separable

(c) $y' = x + y$ is not separable

To solve a separable equation, we follow these steps:

STEP 1 separate the equation

STEP 2 "multiply" by dx and add some snakes

STEP 3 integrate!

STEP 4 solve for y (if you can)

Example 5. Consider the first-order separable differential equation

$$\cos(x)y' = \sin(x)y.$$

Separating, we get:

$$\frac{1}{y}y' = \tan(x)$$

Multiplying by dx, and adding snakes, we obtain

$$\int \frac{1}{y}dy = \int \tan(x)dx$$

This gives

$$\ln|y| = -\ln|\cos(x)| + C$$

and solving for y we find

$$y = C\sec(x).$$

4 First Order Homogeneous Equations

A first order homogeneous differential equation is really a separable equation in disguise.

Definition 4. A first order homogeneous differential equation is a differential equation of the form

$$y' = f(y/x).$$

Example 6. (a) $y' = y/x$ is homogeneous

(b) $y' = (x - y)/(x + y)$ is homogeneous

(c) $y' = x + y$ is not homogeneous

By performing the change of variables

$$y = xz, \quad y' = xz' + z$$

The equation becomes

$$xz' + z = f(z),$$

which is a separable differential equation for z. By solving for z and using $y = xz$, we then obtain y.

Example 7. Consider the differential equation

$$y' = 1 + y/x$$

This equation is homogenous, and doing the z-substitution $y = xz$ and $y' = z + xz'$ we obtain

$$z + xz' = 1 + z,$$

which tells us that $xz' = 1$, and therefore $z' = \frac{1}{x}$. Thus $z = \ln(x) + C$ making

$$y = xz = x\ln(x) + Cx.$$

5 First Order Linear Equations

First order differential linear differential equations are all explicitly solvable also – again up to some integrals.

Definition 5. A first order linear differential equation is a differential equation of the form

$$y' = p(x)y + q(x).$$

Example 8. (a) $y' = x + y$ is linear

(b) $xy' = \sin(x)y + 2xy + 3$ is linear

(c) $y' = y^2$ is not linear

(d) $yy' = x + y$ is not linear

To solve a first-order linear differential equation, our primary method is to find an integrating factor

Definition 6. A first order linear differential equation

$$a(x)y' + b(x)y = c(x)$$

is exact if $a'(x) = b(x)$.

Definition 7. Given a first-order differential equation

$$a(x)y' + b(x)y = c(x),$$

a function $\mu(x)$ satisfying the property that

$$\mu(x)a(x)y' + \mu(x)b(x)y = \mu(x)c(x),$$

is exact is called an integrating factor for the differential equation.

Proposition 2. *An integrating factor for the differential equation*

$$y' = p(x)y + q(x)$$

is

$$\mu(x) = e^{\int p(x)dx}.$$

Example 9. Consider the differential equation

$$y' = x + y$$

An integrating factor for this differential equation is

$$\mu(x) = e^{-\int 1dx} = e^{-x}$$

. Therefore

$$e^{-x}y' = xe^{-x} + e^{-x}y$$

is an exact equation. Rearranging this, we get

$$e^{-x}y' - e^{-x}y = xe^{-x}.$$

By exactness, this becomes

$$(e^{-x}y)' = xe^{-x},$$

and integrating both sides, we obtain

$$e^{-x}y = \int xe^{-x} = -xe^{-x} - e^{-x} + C.$$

Therefore

$$y = -x - 1 + Ce^{x}.$$