Math 307 Section F
Fall 2014
Exam 1
October 20, 2014
Time Limit: 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 11 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

Note also that the LAST PAGE has a couple useful integrals!!

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right.

1. (10 points) (a) (5 points) Find a particular solution of the IVP

$$xy' = -\cos(y), \quad y(1/2) = 0.$$

(b) (5 points) Find the general solution of the differential equation

$$xy' + 2y = \sin(x).$$

Solution 1.

(a) The equation is separable. We separate and integrate, finding

$$\int \sec(y)dy = \int -\frac{1}{x}dx.$$

This tells us that

$$\ln|\sec(y) + \tan(y)| = -\ln|x| + C,$$

and exponentiating we obtain

$$\sec(y) + \tan(y) = B\frac{1}{x}.$$

The initial condition then says

$$\sec(0) + \tan(0) = B\frac{1}{1/2}.$$

Therefore $B = 1/2$, so that our solution is

$$\sec(y) + \tan(y) = \frac{1}{2x}.$$

(b) The equation is linear. We put it in standard form

$$y' = -\frac{2}{x}y + \frac{\sin(x)}{x}.$$

For this we have the integrating factor

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = x^2.$$

The general solution is then

$$y = x^{-2}\int x^2\frac{\sin(x)}{x}dx = x^{-2}\int x\sin(x)dx = x^{-2}(-x\cos(x) + \sin(x) + C).$$

Thus the general solution is

$$y = -\frac{\cos(x)}{x} + \frac{\sin(x)}{x^2} + \frac{C}{x^2}.$$

2. **Multiple Choice Section!**

Directions: The multiple choice section consists of five multiple choice questions. You are NOT required to justify your answer.

(a) (2 points) Which of the following initial value problems could have more than one solution?

A.
$$y' + y^2 = -3xy + \cos(x), \quad y(\pi) = 2\pi^2.$$

B.
$$xy' = 2xy - 3, \quad y(1) = 1.$$

C.
$$y' = y^{1/3}, \quad y(0) = 1.$$

D.
$$y' = |1 - y| + 1, \quad y(0) = 1.$$

(b) (2 points) The equation
$$y' = 3y\frac{xy^2 + yx^2}{x + y}$$
is which type of equation?

A. Linear B. Homogeneous C. Seperable D. Exact

(c) (2 points) What is the largest interval we should expect a unique solution to the initial value problem
$$(3 - x)y' = \sin(x)y + \tan(x), \quad y(\pi) = 0$$

A. $(\pi/2, 3)$ B. $(3, 3\pi/2)$ C. $(\pi/2, 3\pi/2)$ D. $(-\pi/2, \pi/2)$

(d) (2 points) Which of the following statements is TRUE?

A. The general solution of $y' = \frac{1}{y(1-y)(1+y)}$ is $y = \pm\sqrt{\frac{Ce^x}{Ce^x - 1}}$

B. An initial value problem must have at least one solution.

C. If $\mu(x, y)$ is an integrating factor for the equation $M(x, y) + N(x, y)y' = 0$, then $42\mu(x, y)$ is also an integrating factor for the same equation.

D. A family of solutions to $y' = 2x + 2y$ is given by $y = x^2 + y^2 + C$.

(e) (2 points) Which of the following statements is TRUE about the equation
$$y' = y(e^y - e^{-y})(1 - y^2)$$

A. $y = 0$ is a semistable equilibrium

B. $y = 1$ is an unstable equilibrium

C. $y = -1$ is a semistable equilibrium

D. $y = 0$ is a stable equilibrium solution

Solution 2. DCBCA

3. (10 points) (a) (5 points) Show that the following equation is exact. Then find a family of solutions.

$$e^x \sin(y) - 2y\sin(x) + (e^x \cos(y) + 2\cos(x))y' = 0.$$

(b) (5 points) Find an integrating factor for the following differential equation (you do NOT need to solve the equation).

$$3x^2y + 2xy + y^3 + (x^2 + y^2)y' = 0.$$

Solution 3.

(a) In this case, we have

$$M(x,y) = e^x \sin(y) - 2y\sin(x), \quad N(x,y) = e^x \cos(y) + 2\cos(x).$$

Therefore

$$M_y(x,y) = e^x \cos(y) - 2\sin(x), \quad N_x(x,y) = e^x \cos(y) - 2\sin(x).$$

Thus the equation is exact. To solve it, we find $\psi(x,y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Partial integration says

$$\psi(x,y) = \int \psi_x(x,y)\partial x = \int M(x,y)\partial x = \int (e^x \sin(y) - 2y\sin(x))\partial x = e^x \sin(y) + 2y\cos(x) + h(y).$$

From this, we see that

$$\psi_y(x,y) = e^x \cos(y) + 2\cos(x) + h'(y).$$

Comparing this to $N(x,y)$, we obtain that $h'(y) = 0$. Thus without loss of generality, we can take $h(y) = 0$, and $\psi(x,y) = e^x \sin(y) + 2y\cos(x)$. A family of solutions is therefore

$$e^x \sin(y) + 2y\cos(x) = C.$$

(b) We try to find an integrating factor of the form $\mu(x)$. Then

$$\mu(x)(3x^2y + 2xy + y^3) + \mu(x)(x^2 + y^2)y' = 0$$

should be exact. Here

$$M(x,y) = \mu(x)(3x^2y + 2xy + y^3), \quad N(x,y) = \mu(x)(x^2 + y^2).$$

We therefore calculate that

$$M_y(x,y) = \mu(x)(3x^2 + 2x + 3y^2), \quad N(x,y) = \mu'(x)(x^2 + y^2) + \mu(x)(2x).$$

Setting these equal to each other, we get

$$\mu(x)(3x^2 + 2x + 3y^2) = \mu'(x)(x^2 + y^2) + \mu(x)(2x).$$

This simplifies to

$$\mu(x)(3x^2 + 3y^2) = \mu'(x)(x^2 + y^2).$$

Note that we can remove a factor of $x^2 + y^2$ from both sides. Doing so, we obtain

$$\mu(x)3 = \mu'(x).$$

This is a separable equation – a solution is $\mu(x) = e^{3x}$.

4. (10 points) Newton's law of cooling states that the temperature $u(t)$ of an object changes at a rate proportional to $u(t) - T(t)$, where $T(t)$ is the surrounding temperature at time t. To put this another way,

$$u'(t) = -k(u(t) - T(t)),$$

for some constant k which in general depends on the material properties of the object and its surroundings. At noon, a differential equations student fills a bird bath outdoors with water at u_0 degrees F. The temperature outside is given by $T(t) = 65 + 5\cos(\pi t/12)$, where t is given in hours since noon.

(a) Find an equation for $u(t)$ (your answer should involve k and u_0).

(b) After two hours have passed, the temperature of the water in the bath "acclimates", and any terms of the form e^{-kt} in your solution from (a) are very, very small (and may be taken to be zero). With this in mind, write an approximate expression for $u(t)$, for $t > 2$ hours. Does the temperature after two hours depend on the starting temperature u_0?

(c) Suppose that the next day, the differential equations student measures the temperature of the water in the bird bath to be 0.1 degree less than the surrounding temperature. Using your approximation from (b), determine the value of k.

Solution 4.

(a) We must solve the initial value problem

$$u'(t) = -ku(t) + k(65 + 5\cos(\pi t/12), \ u(0) = u_0.$$

The differential equation is linear, and our integrating factor formula tells that an integrating factor is $\mu(t) = e^{kt}$, and that the solution is therefore

$$u(t) = e^{-kt} \int ke^{kt}(65 + 5\cos(\pi t/12)dt = 65 + 5ke^{-kt} \int e^{kt}\cos(\pi t/12)dt.$$

From the integrals on the back page we can evaluate this last integral. Simplifying the result, we find

$$u(t) = 65 + \frac{5k}{k^2 + (\pi/12)^2}\left(k\cos(\pi t/12) + (\pi/12)\sin(\pi t/12)\right) + Ce^{-kt}.$$

To determine the value of the constant C, we use our initial condition. As a result we get $C = u_0 - 65 - \frac{5k^2}{k^2+(\pi/12)^2}$. Thus the final answer is

$$u(t) = 65 + \frac{5k}{k^2 + (\pi/12)^2}\left(k\cos(\pi t/12) + (\pi/12)\sin(\pi t/12)\right) + \left(u_0 - 65 - \frac{5k^2}{k^2 + (\pi/12)^2}\right)e^{-kt}.$$

(b) Taking our answer from (a), we let $e^{-kt} \to 0$, getting

$$u(t) \approx 65 + \frac{5k}{k^2 + (\pi/12)^2}\left(k\cos(\pi t/12) + (\pi/12)\sin(\pi t/12)\right).$$

This does not depend on the initial condition

(c) We have that $u(24) = T(24) - 0.1$, and therefore that

$$0.1 = 5 - \frac{5k^2}{k^2 + (\pi/12)^2}.$$

Solving for k^2, we obtain $k^2 = 49(\pi/12)^2$, and therefore that $k = \sqrt{49}\pi/12 \approx 1.8326$.

5. (10 points)

   (a) (5 points) Find a family of solutions of the following differential equation

$$y' = \frac{x - y}{x + y}.$$

   (b) (5 points) The population of creatures on a radioactive island consists of humans and zombies. Let h denote the fraction of the population which is humans and z denote the fraction of the population which is zombies, so that $h + z = 1$. Humans that come in contact with zombies end up as zombies themselves, so that the rate of change of the zombie population is equal to the number of such contacts. Furthermore, assume both humans and zombies move about the island freely, so that the number of contacts is proportional to the product of h and z, with proportionality constant k. Assuming that 0.99 of the population is human initially, set up (but do not solve) an initial value problem describing the population of zombies as a function of time.

Solution 5.

(a) The equation is homogeneous. Doing the usual substitution $z = y/x$, we obtain

$$z + xz' = \frac{1-z}{1+z}.$$

Therefore

$$xz' = \frac{1-2z-z^2}{1+z}.$$

This equation is separable. Separating and integrating, we find

$$\int \frac{1+z}{1-2z-z^2}dz = \int \frac{1}{x}dx.$$

Therefore

$$-\frac{1}{2}\ln|1-2z-z^2| = \ln|x| + C.$$

Exponentiating both sides, we obtain

$$(1-2z-z^2)^{-1/2} = Bx.$$

Substituting back in for z, this says

$$(1-2(y/x)-(y/x)^2)^{-1/2} = Bx.$$

(b) We have that $z' = kzh$, and since $h = 1 - z$ this becomes $z' = kz(1-z)$. Moreover, since $h(0) = 0.99$ we must have that $z(0) = 0.01$, and our initial value problem is therefore

$$z' = kz(1-z), \quad z(0) = 0.01.$$

SOME HELPFUL INTEGRALS:

$$\int e^{ax}\cos(bx)dx = \frac{a}{a^2+b^2}e^{ax}\cos(bx) + \frac{b}{a^2+b^2}e^{ax}\sin(bx) + C.$$

$$\int e^{ax}\sin(bx)dx = \frac{-b}{a^2+b^2}e^{ax}\cos(bx) + \frac{a}{a^2+b^2}e^{ax}\sin(bx) + C.$$