Math 307 Section F
Spring 2014
Final Exam
June 11, 2014
Time Limit: 1 Hour 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 12 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (a) (5 points) Find the general solution:

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

(b) (5 points) Solve the initial value problem

$$y' = 2y + \cos(t), \quad y(0) = 1/2$$

2. **Propose a Solution Section!**

Directions: The "Propose a Solution" section consists of five linear nonhomogeneous equations. For each of these equations, write down the type of function y (with undetermined coefficients) you would try, in order to get a particular solution. *You do NOT need to solve the equations* For example, if the equation were

$$y'' + 2y' + y = e^t,$$

a *correct answer* would be

$$y = Ae^t,$$

and *incorrect answers* would include

$$y = (At + B)e^t, \quad y = At^2e^{2t}, \quad y = Ae^{3t}, \quad y = A\pi^t$$

Each part is worth 2pts:

(a) (2 points)

$$y'' + 3y' + 2y = t^5e^{4t}$$

(b) (2 points)

$$y'' + 2y' + y = 4t^2e^t + 2e^t$$

(c) (2 points)

$$y'' - 2y' = t - 1$$

(d) (2 points)

$$y'' + 3y' + 2y = (t - 1)e^{-2t}$$

(e) (2 points)

$$y'' - 2y' + y = 2t^2e^t$$

3. (a) (5 points) Find an integrating factor for the equation

$$1 + \sin(xy) + y\cos(xy) + x\cos(xy)y' = 0$$

You do NOT need to solve it

(b) (5 points) Show that the equation

$$e^x + e^{xy} + xye^{xy} + \left(x^2e^{xy} + \cos(y)\right)y' = 0$$

is exact. Then solve it.

4. (10 points) A full tank contains 10 gal of water. Initially, the concentration of dye is 0.5 g/gal. Water with a concentration of dye of 0.1 g/gal flows in at a rate of 1 gal/min. The tank has an outlet at the bottom where 2 gal/min of water flow out. Find the amount of dye contained in the tank when the tank is half full (or half empty, depending on your inclination).

5. (a) (4 points) Find a particular solution of the equation

$$y'' + 2y' + y = e^t \cos(t)$$

(b) (2 points) Find a particular solution of the equation

$$y'' + 2y' + y = e^t \sin(t)$$

(c) (4 points) Find the general solution of the equation

$$y'' + 2y' + y = 3e^t \cos(t) - 2e^t \sin(t)$$

6. (10 points) Recall that the acceleration due to gravity is $g = 9.81$ m/s^2. A 0.3 kg mass is attached to a spring, causing the spring to stretch 1 meter. To to friction, when in motion the mass-spring system experiences a damping force proportional to its current velocity. In particular, experimental observations found that when the mass is moving at 2 meters/sec, it experiences a drag force of 2 Newtons (kg·m/s^2) in the direction opposite to the direction of motion. Suppose that the mass spring system is initially *contracted* 0.5 meters from it's equilibrium state and then released from rest (initial velocity is zero). Determine the position of the mass relative to its equilibrium position $u(t)$ as a function of time. Determine the resultant quasi-amplitude and quasi-frequency.

7. (10 points) Find the Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

8. For each of the following functions $F(s)$, determine the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$

   (a) (5 points)

$$F(s) = \frac{s}{(s-2)^3}$$

   (b) (5 points)

$$F(s) = \frac{2s+3}{s^2+4s+5}$$

9. (10 points) Use Laplace transforms to find the solution to the initial value problem

$$y'' + 3y' + 2y = e^t \sin(2t), \; y(0) = 1, \; y'(0) = -2.$$

10. (10 points) Use Laplace transforms to find the solution to the initial value problem

$$y'' + 2y' + 2y = f(t), \quad y(0) = 0, \ y'(0) = 1,$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ 1 & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi \le t \end{cases}$$

Figure 1: Elementary Laplace Transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$1/s$
e^{at}	$\frac{1}{s-a}$
t^n, $n \geq 0$ integer	$\frac{n!}{s^{n+1}}$
t^p, $p \geq 0$ real	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
$t^n e^{at}$ $n \geq 0$ integer	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F(s/c)$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$