Math 307 Section F
Spring 2014
Final Exam
June 11, 2014
Time Limit: 1 Hour 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 23 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (a) (5 points) Find the general solution:

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

(b) (5 points) Solve the initial value problem

$$y' = 2y + \cos(t), \quad y(0) = 1/2$$

Solution 1.

(a) If we divide by x^2 on both sides, we get

$$y' = 1 + (y/x) + (y/x)^2,$$

which is a homogeneous equation. To solve this, we use the substitution $z = y/x$. This means that $y' = z + xz'$ and therefore

$$z + xz' = 1 + z + z^2.$$

This simplifies to

$$xz' = 1 + z^2.$$

This equation is separable. Separating and integrating, we obtain

$$\tan^{-1}(z) = \ln|x| + C.$$

Solving for z, this gives

$$z = \tan(\ln|x| + C)$$

Then substituting $z = y/x$ and solving for y, we obtain

$$y = x\tan(\ln|x| + C).$$

(b) The equation is linear, since it is of the form $y' = p(t)y + q(t)$ for $p(t) = 2$ and $q(t) = \cos(t)$. Our integrating factor formula for first-order linear equations gives us the integrating factor

$$\mu(t) = e^{\int -p(t)dt} = e^{\int -2dt} = e^{-2t}.$$

Then the general solution is given by

$$y = \frac{1}{\mu(t)}\int \mu(t)q(t)dt = e^{2t}\int e^{-2t}\cos(t)dt.$$

To do this last integral, we can use the equation

$$\int e^{at}\cos(bt)dt = \frac{a}{a^2+b^2}e^{at}\cos(bt) + \frac{b}{a^2+b^2}e^{bt}\cos(bt) + C$$

with $a = -2, b = 1$. The general solution is therefore

$$y = \frac{-2}{5}\cos(t) + \frac{1}{5}\sin(t) + Ce^{2t}.$$

The solution of the initial value problem is therefore

$$y = \frac{-2}{5}\cos(t) + \frac{1}{5}\sin(t) + \frac{9}{10}e^{2t}.$$

2. **Propose a Solution Section!**

Directions: The "Propose a Solution" section consists of five linear nonhomogeneous equations. For each of these equations, write down the type of function y (with undetermined coefficients) you would try, in order to get a particular solution. *You do NOT need to solve the equations* For example, if the equation were

$$y'' + 2y' + y = e^t,$$

a *correct answer* would be

$$y = Ae^t,$$

and *incorrect answers* would include

$$y = (At + B)e^t, \quad y = At^2e^{2t}, \quad y = Ae^{3t}, \quad y = A\pi^t$$

Each part is worth 2pts:

(a) (2 points)

$$y'' + 3y' + 2y = t^5e^{4t}$$

(b) (2 points)

$$y'' + 2y' + y = 4t^2e^t + 2e^t$$

(c) (2 points)

$$y'' - 2y' = t - 1$$

(d) (2 points)

$$y'' + 3y' + 2y = (t - 1)e^{-2t}$$

(e) (2 points)

$$y'' - 2y' + y = 2t^2e^t$$

Solution 2.

(a)
$$y_p = (A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 + A_5 t^5)e^{4t}.$$

(b)
$$y_p = (A_0 + A_1 t + A_2 t^2)e^{t}.$$

(c)
$$y_p = (A_0 t + A_1 t^2).$$

(d)
$$y_p = (A_0 t + A_1 t^2)e^{-2t}.$$

(e)
$$y_p = (A_0 t^2 + A_1 t^3 + A_2 t^4)e^{t}.$$

3. (a) (5 points) Find an integrating factor for the equation

$$1 + \sin(xy) + y\cos(xy) + x\cos(xy)y' = 0$$

You do NOT need to solve it

(b) (5 points) Show that the equation

$$e^x + e^{xy} + xye^{xy} + \left(x^2e^{xy} + \cos(y)\right)y' = 0$$

is exact. Then solve it.

Solution 3.

(a) We propose an integrating factor of the form $\mu(x,y) = \mu(x)$. Then

$$\overbrace{\mu(x) + \mu(x)\sin(xy) + y\mu(x)\cos(xy)}^{M(x,y)} + \overbrace{x\mu(x)\cos(xy)}^{N(x,y)}\, y' = 0$$

should be exact. This means that $M_y(x,y) = N_x(x,y)$ and therefore that

$$\mu(x)x\cos(xy) + \mu(x)\cos(xy) - xy\mu(x)\sin(xy) = \mu(x)\cos(xy) + x\mu'(x)\cos(xy) - xy\mu(x)\sin(xy).$$

This simplifies to

$$\mu(x)x\cos(xy) = x\mu'(x)\cos(xy).$$

Then we can divide by $x\cos(xy)$ to obtain

$$\mu(x) = \mu'(x).$$

This equation is separable – a solution is $\mu(x) = e^x$, which is an integrating factor.

(b) In this equation

$$M(x,y) = e^x + e^{xy} + xye^{xy}, \quad N(x,y) = x^2e^{xy} + \cos(y).$$

Therefore

$$M_y(x,y) = 2xe^{xy} + x^2ye^{xy}, \quad N_x(x,y) = 2xe^{xy} + x^2ye^{xy},$$

which shows that $M_y = N_x$, so that the equation is exact. To solve it, we find $\psi(x,y)$ such that $\psi_x = M$ and $\psi_y = N$. The latter equation says

$$\psi(x,y) = \int \psi_y(x,y)\partial y = \int N(x,y)\partial y = \int (x^2e^{xy} + \cos(y))\partial y = xe^{xy} + \sin(y) + g(x).$$

Taking the partial derivative of this expression with respect to x, we see that

$$\psi_x(x,y) = e^{xy} + xye^{xy} + g'(x).$$

Then since $\psi_x = M$, this says that

$$e^{xy} + xye^{xy} + g'(x) = e^x + e^{xy} + xye^{xy}.$$

This simplifies to $g'(x) = e^x$. Therefore we can take $g(x) = e^x$, and

$$\psi(x,y) = xe^{xy} + \sin(y) + e^x.$$

Then the family of solutions we want is obtained by setting $\psi = C$, eg.

$$xe^{xy} + \sin(y) + e^x = C.$$

4. (10 points) A full tank contains 10 gal of water. Initially, the concentration of dye is 0.5 g/gal. Water with a concentration of dye of 0.1 g/gal flows in at a rate of 1 gal/min. The tank has an outlet at the bottom where 2 gal/min of water flow out. Find the amount of dye contained in the tank when the tank is half full (or half empty, depending on your inclination).

Solution 4.

$$\frac{dV}{dt} = 1 - 2 = -1.$$

Since $V(0) = 10$, it follows that $V(t) = 10 - t$. The mass of dye in the tank is governed by the differential equation

$$\frac{dm}{dt} = (0.1) \cdot 1 - \frac{m}{V} \cdot 2.$$

Therefore

$$\frac{dm}{dt} = 0.1 - \frac{2}{10-t} m.$$

This differential equation is linear, and we can solve it using an integrating factor. Using the usual formula, the integrating factor is found to be $\mu(t) = (10-t)^{-2}$, and the solution is

$$m(t) = (10-t)^2 \int \frac{0.1}{(10-t)^2} dt = 0.1(10-t) + C(10-t)^2.$$

Moreover, since the initial concentration of dye in the tank is 0.5 g/gal, the mass of dye in the tank initially is 5 grams. Therefore $m(0) = 5$. This means that $5 = 1 + 100C$, making $C = 2/50 = 0.04$. Therefore

$$m(t) = 0.1(10-t) + 0.04(10-t)^2 = 0.04t^2 - 0.9t + 5$$

The tank is half-full when $t = 5$. Thus the answer we are looking for is $m(5) = 1.5$ grams.

5. (a) (4 points) Find a particular solution of the equation

$$y'' + 2y' + y = e^t \cos(t)$$

(b) (2 points) Find a particular solution of the equation

$$y'' + 2y' + y = e^t \sin(t)$$

(c) (4 points) Find the general solution of the equation

$$y'' + 2y' + y = 3e^t \cos(t) - 2e^t \sin(t)$$

Solution 5.

(a) First we think about the "squigglified" equation

$$\widetilde{y}'' + 2\widetilde{y}' + \widetilde{y} = e^{(1+i)t}.$$

Using the method of undetermined coefficients, we propose a solution of the form

$$\widetilde{y_p} = Ae^{(1+i)t}.$$

Plugging this in to the squiggly equation, we find

$$(3+4i)Ae^{(1+i)t} = e^{(1+i)t}.$$

This says that

$$A = \frac{1}{3+4i} = \frac{3}{25} + \frac{-4}{25}i.$$

Therefore

$$\widetilde{y_p} = \left(\frac{3}{25} + \frac{-4}{25}i\right) e^{(1+i)t}.$$

Using Euler and factoring this out, we get

$$\widetilde{y_p} = \left[\frac{3}{25}e^t \cos(t) + \frac{4}{25}e^t \sin(t)\right] + i\left[\frac{-4}{25}e^t \cos(t) + \frac{3}{25}e^t \sin(t)\right].$$

A particular solution to the original equation is therefore

$$y_p = \mathrm{Re}(\widetilde{y_p}) = \frac{3}{25}e^t \cos(t) + \frac{4}{25}e^t \sin(t).$$

(b) Using the squiggly work of the previous part, we get a particular solution

$$y_p = \mathrm{Im}(\widetilde{y_p}) = \frac{-4}{25}e^t \cos(t) + \frac{3}{25}e^t \sin(t).$$

(c) By taking a linear combination of the particular solutions y_{pa} in (a) and y_{pb} (b), we can obtain a particular solution of the equation in (c):

$$y_p = 3y_{pa} - 2y_{pb} = \frac{17}{25}e^t \cos(t) + \frac{6}{25}e^t \sin(t).$$

The general solution is obtained by adjoining the general solution of the corresponding homogeneous equation. Thus the general solution is

$$y = \frac{17}{25}e^t \cos(t) + \frac{6}{25}e^t \sin(t) + (At+B)e^{-t}.$$

6. (10 points) Recall that the acceleration due to gravity is $g = 9.81$ m/s^2. A 0.3 kg mass is attached to a spring, causing the spring to stretch 1 meter. To to friction, when in motion the mass-spring system experiences a damping force proportional to its current velocity. In particular, experimental observations found that when the mass is moving at 2 meters/sec, it experiences a drag force of 2 Newtons (kg·m/s^2) in the direction opposite to the direction of motion. Suppose that the mass spring system is initially *contracted* 0.5 meters from it's equilibrium state and then released from rest (initial velocity is zero). Determine the position of the mass relative to its equilibrium position $u(t)$ as a function of time. Determine the resultant quasi-amplitude and quasi-frequency.

Solution 6.

We calculate $m = 0.3$ kg , $k = (0.3) \cdot (9.81) = 2.943$ N/m and $\gamma = 2/2 = 1$ Ns/m. Furthermore $u(0) = -0.5$ and $u'(0) = 0$, so the initial value problem governing the motion of the spring is

$$0.3u'' + u' + 2.943u = 0.$$

The characteristic polynomial has roots $-1.667 \pm 2.652i$. Therefore the general solution is

$$u(t) = Ae^{-1.667t}\cos(2.652t) + Be^{-1.667t}\sin(2.652t).$$

From this, we calculate that

$$u(0) = A, \quad u'(0) = -1.667A + 2.652B.$$

The initial conditions then tell us $A = -0.5$ and $B = -0.314$. Thus we find

$$u(t) = -0.5e^{-1.667t}\cos(2.652t) - 0.314e^{-1.667t}\sin(2.652t).$$

From this we see that the quasi-frequency is

$$\omega = 2.652$$

and that the quasi-amplitude is

$$R = \sqrt{(-0.5)^2 + (-0.314)^2} = 0.590.$$

7. (10 points) Find the Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

Solution 7. We can write $f(t)$ in step-function form as

$$f(t) = t + (1 - t)u_1(t).$$

Note that alternatively we could write

$$f(t) = tu_0(t) + (1 - t)u_1(t).$$

Either expression will lead to the *same* Laplace transform, since the Laplace transform only "sees" what the function looks like for positive values of t. To take the Laplace transform, we then use the identity

$$\mathcal{L}\left\{f(t)u_c(t)\right\} = e^{-cs}\mathcal{L}\left\{f(t+c)\right\}.$$

Therefore

$$\mathcal{L}\left\{(1-t)u_1(t)\right\} = e^{-s}\mathcal{L}\left\{(1-(t+1))\right\} = e^{-s}\mathcal{L}\left\{-t\right\} = -e^{-s}\frac{1}{s^2}.$$

Since $\mathcal{L}\left\{t\right\} = 1/s^2$, we conclude

$$\mathcal{L}\left\{f(t)\right\} = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}.$$

8. For each of the following functions $F(s)$, determine the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$

(a) (5 points)

$$F(s) = \frac{s}{(s-2)^3}$$

(b) (5 points)

$$F(s) = \frac{2s+3}{s^2+4s+5}$$

Solution 8.

(a) To do this problem, we first find the PFD form of $F(s)$. We have

$$\frac{s}{(s-2)^3} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3}.$$

Solving for A, B, C, we find $A = 0$, $B = 1$, $C = 2$. Therefore

$$F(s) = \frac{1}{(s-2)^2} + \frac{2}{(s-2)^3}.$$

Thus

$$f(t) = te^{2t} + t^2e^{2t}.$$

(b) The rational function $F(s)$ is already decomposed. Completing the square in the denominator, we find

$$F(s) = \frac{2s+3}{(s+2)^2+1}.$$

We would like the numerator to involve $s+2$ also. We can accomplish this with a couple algebraic manipulations

$$\frac{2s+3}{(s+2)^2+1} = \frac{2(s+2)-1}{(s+2)^2+1} = 2\frac{s+2}{(s+2)^2+1} - \frac{1}{(s+2)^2+1}.$$

Thus

$$f(t) = 2e^{-2t}\cos(t) - e^{-2t}\sin(t).$$

9. (10 points) Use Laplace transforms to find the solution to the initial value problem

$$y'' + 3y' + 2y = e^t \sin(2t), \ y(0) = 1, \ y'(0) = -2.$$

Solution 9. Taking the Laplace transform of both sides, we obtain

$$(s^2 + 3s + 2)\mathcal{L}\{y\} - s - 1 = \frac{2}{(s-1)^2 + 4}.$$

Solving for $\mathcal{L}\{y\}$, we obtain

$$\mathcal{L}\{y\} = \frac{2}{(s^2 + 3s + 2)((s-1)^2 + 4)} + \frac{s+1}{s^2 + 3s + 2}.$$

We factor $s^2 + 3s + 2 = (s+2)(s+1)$, so this says

$$\mathcal{L}\{y\} = \frac{2}{(s^2 + 3s + 2)((s-1)^2 + 4)} + \frac{1}{s+2}.$$

Now we need to do the PFD of the first rational function on the right hand side. We have

$$\frac{2}{(s^2 + 3s + 2)((s-1)^2 + 4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs + D}{(s-1)^2 + 4}.$$

Solving for A, B, C, D we find $A = 1/4$, $B = -2/13$, $C = -5/52$, $D = 7/52$, and therefore

$$\mathcal{L}\{y\} = \frac{1/4}{s+1} + \frac{11/13}{s+2} + \frac{1}{52}\frac{-5s + 7}{(s-1)^2 + 4}.$$

To invert, we do the usual manipulation on the term with the completed square in the denominator, obtaining

$$\frac{-5s + 7}{(s-1)^2 + 4} = \frac{-5(s-1) + 2}{(s-1)^2 + 4} = -5\frac{s-1}{(s-1)^2 + 4} + \frac{2}{(s-1)^2 + 4}.$$

Therefore we obtain

$$\mathcal{L}\{y\} = \frac{1/4}{s+1} + \frac{11/13}{s+2} - \frac{5}{52}\frac{s-1}{(s-1)^2 + 4} + \frac{1}{52}\frac{2}{(s-1)^2 + 4}.$$

Thus

$$y = \frac{1}{4}e^{-t} + \frac{11}{13}e^{-2t} - \frac{5}{52}e^t \cos(2t) + \frac{1}{52}e^t \sin(2t).$$

10. (10 points) Use Laplace transforms to find the solution to the initial value problem

$$y'' + 2y' + 2y = f(t), \quad y(0) = 0, \; y'(0) = 1,$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < \pi \\ 1 & \text{if } \pi \leq t < 2\pi \\ 0 & \text{if } 2\pi \leq t \end{cases}$$

Solution 10. We first convert $f(t)$ to step function form, getting

$$f(t) = u_\pi(t) - u_{2\pi}(t).$$

Therefore

$$\mathcal{L}\{f(t)\} = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}.$$

Taking the Laplace transform of both sides of the original differential equation, it follows that

$$(s^2 + 2s + 2)\mathcal{L}\{y\} - 1 = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}.$$

Thus

$$\mathcal{L}\{y\} = \frac{1}{s(s^2 + 2s + 2)}e^{-\pi s} - \frac{1}{s(s^2 + 2s + 2)}e^{-2\pi s} + \frac{1}{s^2 + 2s + 2}.$$

Defining $F(s) = \frac{1}{s(s^2+2s+2)}$, we can rewrite this as

$$\mathcal{L}\{y\} = F(s)e^{-\pi s} - F(s)e^{-2\pi s} + \frac{1}{s^2 + 2s + 2}.$$

Now doing a PFD of the rational function $F(s)$, we obtain

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}.$$

Solving for A, B, C, we get $A = 1/2$, $B = -1/2$ and $C = -1$. Therefore

$$F(s) = \frac{1/2}{s} + \frac{(-1/2)s - 1}{s^2 + 2s + 2}.$$

Completing the square and the usual shenanigans, this gives

$$F(s) = \frac{1/2}{s} - \frac{1}{2}\frac{(s+1)}{(s+1)^2 + 1} - \frac{1}{2}\frac{1}{(s+1)^2 + 1}.$$

Thus $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos(t) - \frac{1}{2}e^{-t}\sin(t).$$

Then since

$$\mathcal{L}\{y\} = F(s)e^{-\pi s} - F(s)e^{-2\pi s} + \frac{1}{s^2 + 2s + 2}$$

and also

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1},$$

it follows that

$$y = f(t - \pi)u_\pi(t) - f(t - 2\pi)u_{2\pi}(t) + e^{-t}\sin(t)$$

More explicitly,

$$\begin{aligned} y = &\left[\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - \frac{1}{2}e^{-(t-\pi)}\sin(t-\pi)\right]u_\pi(t) \\ &- \left[\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\cos(t-2\pi) - \frac{1}{2}e^{-(t-2\pi)}\sin(t-2\pi)\right]u_{2\pi}(t) \\ &+ e^{-t}\sin(t). \end{aligned}$$

Figure 1: Elementary Laplace Transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$1/s$
e^{at}	$\frac{1}{s-a}$
t^n, $n \geq 0$ integer	$\frac{n!}{s^{n+1}}$
t^p, $p \geq 0$ real	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
$t^n e^{at}$ $n \geq 0$ integer	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F(s/c)$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$(-t)^nf(t)$	$F^{(n)}(s)$