Math 307 Lecture 1

Introducing Differential Equations!

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Today!

Plan for today:

- What is a differential equation?
- First order differential equations
- Separable and homogeneous equations

Next time:

- Linear equations
- Method: Integrating factors
- Method: Variation of parameters

Outline

What's a Differential Equation?

Question

What is a differential equation?

Definition

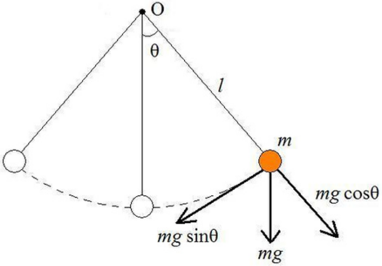
A *differential equation* is mathematical expression describing a relationship between a function and its derivatives

Before going further we should think about:

- Examples of differential equations
- Why we care about differential equations
- The scope of our study of diff. equations in MATH 307

Example Diff. Eqn: Motion of a Rigid Pendulum

Figure: A physics-type picture you've probably seen before



- Newton's second law: $\tau = I\frac{d^2\theta}{dt^2}$
- Torque: $\tau = mgl \sin\theta$
- Moment of inertia: $I = ml^2$
- We get a differential equation!

$$\frac{d^2\theta}{dt^2} = \frac{mg}{I}\sin\theta$$

Example Diff. Eqn: Compound interest

Figure: A traditional celebration of compound interest as demonstrated by the notable entrepreneur Scrooge Mc. Duck

- For continuously compounded interest

$$\frac{dS}{dt} = rS$$

- S is invested capital
- r is interest rate
- This is a differential equation!
- Solution is $S(t) = S_0 e^{rt}$ (How do we get this?)

Example Diff. Eqn: Falling with air drag

Figure: Diving to the earth from the stratosphere is probably more fun than differential equations. Maybe?

- Newton's second law: $F = ma$
- Using a linear drag model

$$m\frac{d^2y}{dt^t} = -mg + k\frac{dy}{dt}$$

- y is your height
- g is gravitational acceleration
- k is a drag coefficient
- How can we solve this equation to get y?

Example Diff. Eqn: Fluid flow in one dimension

Figure: A fluid flow is as cool as it is complicated! Below is an example of what are called Von Karman vortices. (2-dim, so not covered in this course)

- Goal: find velocity of the fluid $u = u(x, t)$
- x is position in the fluid
- t is time
- p is pressure
- ρ is density

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{d^2 p}{dx^2}$$

- It's a *partial differential equation* because it has partial derivatives

The What and Wh

Student: Why should we learn about differential equations?

Wizard: Because they naturally come up all over the place!

Student: What kinds of differential equations will we learn about?

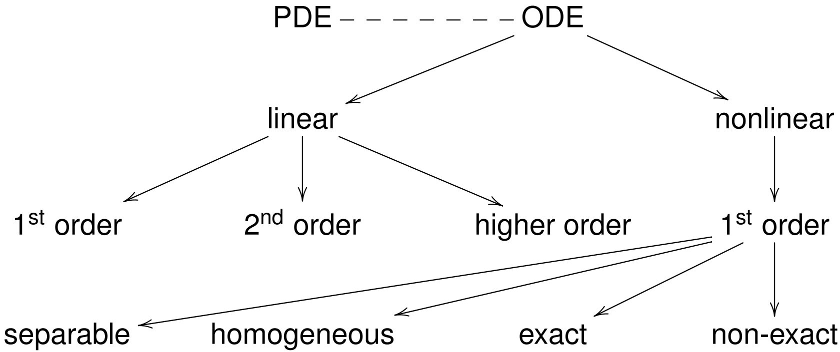
Wizard: There's just too much to learn! We will focus on what are called first and second order equations.

Student: How hard is it to solve a differential equation?

Wizard: Differential equations, even first and second order ones, can be really hard to solve! Our goal: learn to identify ones which are easy to solve and how

Classification of Differential Equations

Figure: A MATH 307 perspective of the "types" of Diff. Eqns



First Order Equations

Definition

A first order ordinary differential equation (ODE) is an equation of the form

$$\frac{dy}{dt} = f(t, y)$$

where f is a function of the two variables t and y.

- Our goal is to find **solutions** to first order differential equations
- Algebraically: find function $y(t)$ satisfying the above equation
- Geometrically: find finding a curve matching a **slope field**

Solutions to First-Order equations

- Almost always, an ODE $y' = f(t, y)$ will have lots of different solutions
- However, for nice ODEs, may be exactly one solution satisfying $y(a) = b$
- The additional constraint $y(a) = b$ is called a **initial condition**
- The differential equation $y' = f(t, y)$ combined with the constraint $y(a) = b$ is called an **initial value problem** (IVP)

Slope Fields

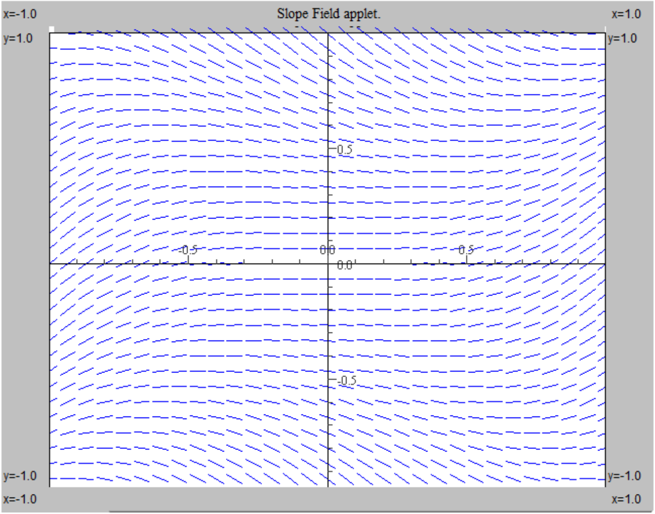
- Slope field is a geometric representation of a first-order ODE
$$\frac{dy}{dt} = f(t, y)$$

PROCESS:

1. Make a "grid" of points in the x, y-plane
2. At each grid point (a, b), draw a dash with slope $f(x, y)$
3. This process creates a **vector field** representing the ODE

- Let's look at an example!

Slope Field Example: $\frac{dy}{dx} = x^2 - y^2$



Slope Fields

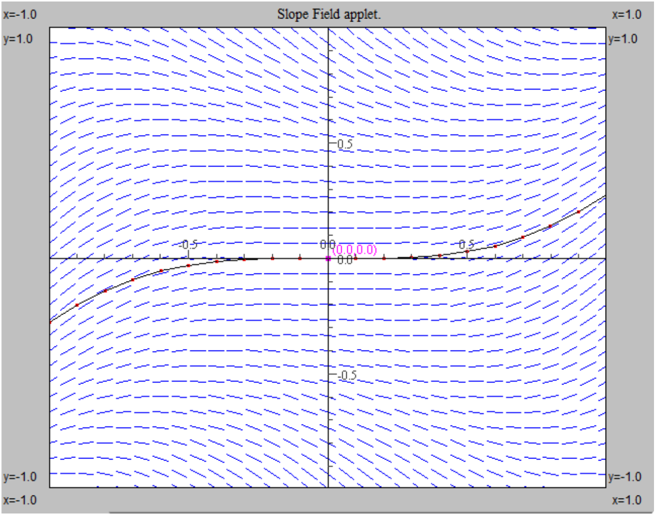
- Solutions to the ODE fit naturally in this picture!

PROCESS:

1. Imagine the slope field as currents in an ocean
2. Put a boat at a point (a, b)
3. Let the boat (quasi-statically) follow the flow

- The path it traces out forms a solution to the ODE
- Example time, how exciting!!

Slope Field Example: Solution satisfying $y(0) = 0$



Slope Fields

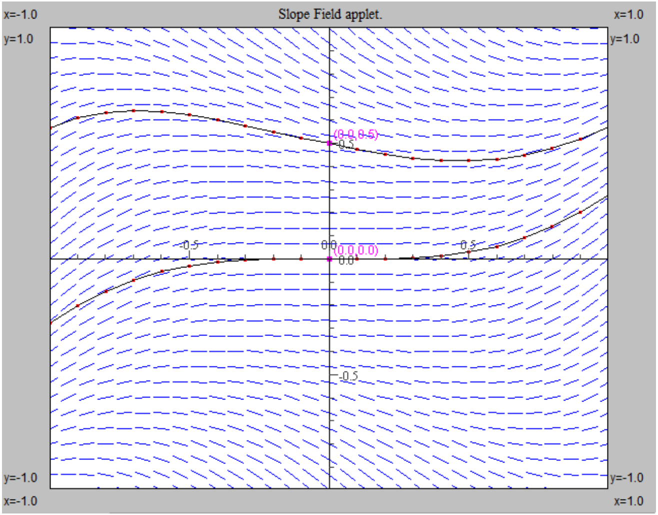
- Of course, where your boat goes depends on where it starts!
- Last time it started at $(0, 0)$, and we got a solution to the IVP

$$y' = x^2 - y^2, \;\; y(0) = 0$$

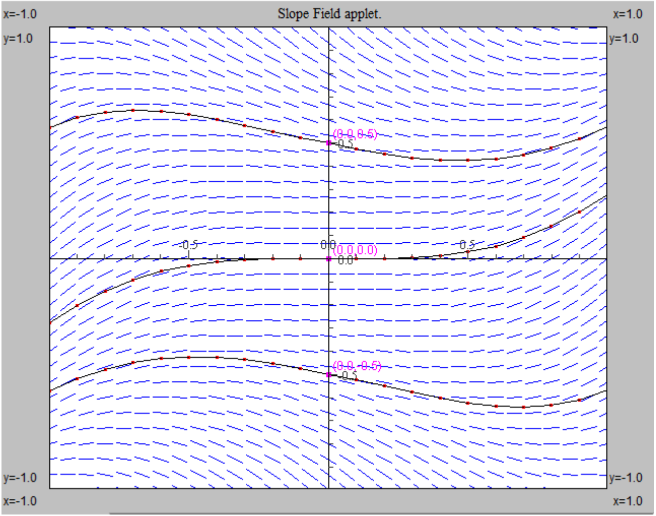
- So if we put our boat at $(0, \pm 0.5)$, we should get a solution to

$$y' = x^2 - y^2, \;\; y(0) = \pm 0.5$$

Slope Field Example: Solution satisfying $y(0) = 0.5$



Slope Field Example: Solution satisfying $y(0) = -0.5$



Separable Equation

Definition

A first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called *separable* if $f(x, y) = g(x)h(y)$ for some functions g, h

Examples:

- $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$
- $\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$
- $y' = (e^{-x} - e^x)/(3 + 4y)$
- $\sin(2x)dx + \cos(3y)dy = 0$

Solving a Separable Equation

Question

How can we solve a separable equation?

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}dy = g(x)dx$$

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

... finish by solving for y

An Example Worked out

Example

Find a solution to the differential equation $y' = (1-2x)y^2$ satisfying the initial condition $y(0) = -1/6$.

$$\frac{1}{y^2}y' = (1-2x)$$

$$\int \frac{1}{y^2}dy = \int (1-2x)dx$$

$$-\frac{1}{y} = x - x^2 + C$$

$$y = \frac{-1}{x - x^2 + C}$$

$y(0) = -1/6$ implies $C = 6$. Hence $y = \frac{-1}{x-x^2+6}$.

Homogeneous Equations

Definition

A homogeneous equation is a first order differential equation of the form

$$\frac{dy}{dx} = f(x,y),$$

where $f(x,y) = g(y/x)$ for some function g.

- Homogeneous equations have "scale invariant" slope fields: if you zoom out, the slope field looks the same!
- Homogeneous equations are separable equations in disguise!

Examples:

- $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$
- $y' = \frac{3y^2-x^2}{2xy}$

Solving Homogeneous Equations

Steps to solve:

- Do enough algebra to write $\frac{dy}{dx} = g(y/x)$
- Define a new variable $z = y/x$
- Since $xz = y$, implicit differentiation says

$$z + x\frac{dz}{dx} = \frac{dy}{dx}$$

- Plugging back into the original DE, we get a separable equation

$$z + x\frac{dz}{dx} = g(z)$$

- Solve ths separable equation for z and use $y = xz$ to WIN

An Example Worked Out

Question

Find a solution to the differential equation $y' = \frac{x^2+xy+y^2}{x^2}$ satisfying the initial condition $y(1) = 0$.

- Notice that $y' = 1 + \frac{y}{x} + \frac{y^2}{x^2} = g(y/x)$ for $g(z) = 1 + z + z^2$
- If we set $z = y/x$, then we find $z + x\frac{dz}{dx} = 1 + z + z^2$

$$\frac{dz}{dx} = \frac{1+z^2}{x}$$
$$\frac{1}{1+z^2}dz = \frac{1}{x}dx$$
$$\int \frac{1}{1+z^2}dz = \int \frac{1}{x}dx$$

An Example Worked Out ∼ Continued

$$\begin{aligned} \arctan(z) &= \ln|x| + C \\ z &= \tan(\ln|x| + C) \\ y &= xz = x\tan(\ln|x| + C) \end{aligned}$$

- Since $y(1) = 0$, we must have $0 = 1\tan(\ln|1| + C) = \tan(C)$.
- This tells us $C = 0$. Hence the solution we want is

$$y(x) = x\tan(\ln|x|)$$

Summary!

What we did today:

- We learned what a differential equation is and why we should care
- We caught a glimpse of first order differential equations
- We learned how to solve separable and homogeneous equations

Plan for next time:

- Linear equations
- Method: Integrating factors
- Method: Variation of parameters