Math 307 Lecture 11

Second-Order Homogeneous Linear ODEs with Constant Coefficients II

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Today!

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

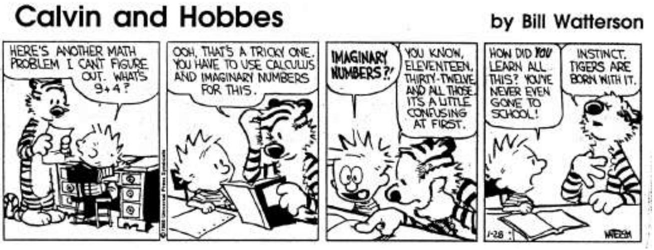
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

Outline

What are imaginary numbers?

- An *imaginary number* is any real number multiplied by $\sqrt{-1}$
- Usually denote this by i
- **examples:** $13i$, $\sqrt{2}i$, $-4i$, πi

Figure: Imaginary numbers are very real to tigers



What are complex numbers?

Definition

A *complex number* z is a number of the form

$$z = \overbrace{a}^{\text{real part}} + \overbrace{b}^{\text{imaginary part}} i$$

where a and b are any real numbers.

- Any real number is a complex number also
- Any imaginary number is a complex number also
- $2 + 3i$, $-\sqrt{7} - \frac{1}{2}i$ and $4 + 2\pi i$ are complex numbers too

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Addition and Subtraction:

- To add complex things, just add real and imaginary parts.
- For example

$$(2+3i)+(4+2\pi i)=6+(3+2\pi)i$$

- Similar story for subtraction...

$$(2+3i)-(4+2\pi i)=-2+(3-2\pi)i$$

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Multiplication:

- To multiply complex things, we have to “foil", remembering that $i^2 = -1$
- For example

$$
\begin{aligned}
(2+3i)\cdot(4+2\pi i) &= 2\cdot 4 + 2\cdot 2\pi i + 3i\cdot 4 + 3i\cdot 2\pi i \\
&= 8 + 4\pi i + 12i + 6\pi i^2 \\
&= 8 + 4\pi i + 12i - 6\pi \\
&= (8-6\pi) + (12+4\pi)i
\end{aligned}
$$

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Inverses:

- We calculate the inverse of a complex number $z = a + bi$ by using the complex conjugate trick
- Complex conjugate is $z^* = a - bi$

$$\begin{aligned}\frac{1}{a+bi} &= \frac{1}{a+bi}\cdot 1 = \frac{1}{a+bi}\cdot\frac{(a-bi)}{(a-bi)}\\ &= \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\end{aligned}$$

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Division:

- We divide by multiplying by the inverse
- Alternatively we apply complex conjugate trick
- For example

$$\frac{4+2\pi i}{2+3i} = \frac{4+2\pi i}{2+3i} \cdot 1 = \frac{4+2\pi i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}$$
$$= \frac{(4+2\pi i)\cdot(2-3i)}{(2+3i)\cdot(2-3i)} = \frac{8+6\pi}{13} + \frac{4\pi-12}{13}i$$

Try it Yourself!

Have a go at the following calculations:

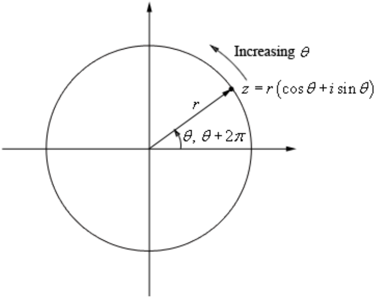
- $$(-3+2i)+(4-6i) = ?$$

- $$(3+i)*(2-i) = ?$$

- $$\frac{1+2i}{3-7i} = ?$$

Complex numbers as vectors

Figure: A complex number is a vector in the plane



- We can "visualize" a complex number $z = x + iy$ as a *vector*
- The tip of the vector is put at the point (x, y)
- The base of the vector is put at the origin
- Using polar coordinates, we can write

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Euler's Definition

Definition

If θ is a real number, then we *define*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Theorem

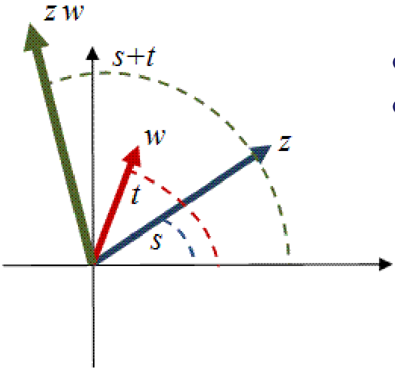
This definition does not break anything.

- Q: What do we mean by this?
- A: $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$
- Also, this makes sense in a power-series sort of way:

$$\sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}i = \cos(\theta)+\sin(\theta)i = e^{i\theta}$$

Complex numbers as vectors

Figure: What does multiplication do to vectors?



- Take a complex number $x + iy$
- Convert (x, y) to polar:
  $r = \sqrt{x^2 + y^2}$
  $\theta = \tan^{-1}(y/x)$
- Then $x + iy = re^{i\theta}$ also!
- If $z = r_0 e^{is}$ and $w = r_1 e^{it}$ then

$$zw = r_0 r_1 e^{i(s+t)}$$

Angles add! (see figure)

A few more definitions

Figure: SMBC

Image redacted

- Let $z = re^{i\theta}$ be a complex number
- θ is called the *argument* of z
- r is called the *modulus* of z and is sometimes denoted as $|z|$
- easy exercise: show $r^2 = z \cdot z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

Our First Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$y'' - 2y' + 2y = 0$$

- How do we find the general solution?
- First we bluff and say we already know our solution: $y = e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2 e^{rt}$

Our First Example

- Then since y is a solution

$$\begin{aligned} 0 &= y'' - 2y' + 2y \\ &= r^2 e^{rt} - 2re^{rt} + 2e^{rt} \\ &= (r^2 - 2r + 2)e^{rt} \end{aligned}$$

- Therefore $r^2 - 2r + 2 = 0$
- Roots of the polynomial $r^2 - 2r + 2$ are $1 \pm i$
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

Our First Example

- We only want real-valued solutions. So look at the general solution (obtained by superposition principal):

$$y = Ae^{(1+i)t} + Be^{(1-i)t}$$

- Using Euler's formula

$$\begin{aligned} y &= Ae^t\cos(t) + iAe^t\sin(t) + Be^t\cos(t) - iBe^t\sin(t) \\ &= (A+B)e^t\cos(t) + i(A-B)e^t\sin(t) \end{aligned}$$

- Here A and B should also be allowed to be *complex* and must be chosen to make y real

Our First Example

- Set $C = A + B$ and $D = i(A - B)$
- Then C and D are arbitrary constants and the general solution is

$$y = Ce^t \cos(t) + De^t \sin(t)$$

- **Note:** C and D must be real to make y real...

General Case

- Consider the equation

$$y'' + ay' + by = 0$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta i$
- Then it will also have the root $\alpha - \beta i$
- Therefore the general (complex-valued) solution will look like

$$y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}$$

General Case

- Euler's formula will then give

$$y = (A + B)e^{\alpha t}\cos(\beta t) + (A - B)e^{\alpha t}\sin(\beta t)i$$

- Setting $C = A + B$ and $D = (A - B)i$, the (real) general solution is

$$y = Ce^{\alpha t}\cos(\beta t) + De^{\alpha t}\sin(\beta t)$$

- Again C and D are completely arbitrary, but must be real-valued for y to be real.

Try it Yourself!

Find the general solutions:

- $$y'' + 2y' - 8y = 0$$

- $$y'' + 6y' + 13y = 0$$

- $$y'' + 2y' + 1.25y = 0$$

Review!

Today:

- Complex numbers and Euler's definition
- What happens when the characteristic polynomial has complex roots

Next time:

- What happens when the characteristic polynomial has repeated roots