Math 307 Lecture 12

Second-Order Homogeneous Linear ODEs with Constant Coefficients III

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Today!

Last time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

Next time:

- 2nd-Order Nonhomogeneous Linear ODEs. with Constant Coefficients
- Method of Undetermined Coefficients

Outline

Review: What do we know?

Question

Do we know how to solve 2nd-order linear homogeneous ODEs with constant coefficients yet?

- In other words, do we know the general solution to

$$ay'' + by' + cy = 0,$$

(with $a > 0$) for any choice of a, b, c?
- Almost!
- We try a solution of the form $= e^{rt}$
- For this to work, r must be a root of the *characteristic equation*

$$ar^2 + br + c = 0$$

Review: If the roots are distinct and real...

- Suppose the two roots of the characteristic equation are r_1 and r_2
- If r_1, r_2 are *distinct* and *real*, then we have two solutions right away!
- Namely $y = e^{r_1 t}$ and $y = e^{r_2 t}$ are solutions
- By the *superposition principal*, we actually have a two-parameter family of solutions:

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- This turns out to be the general solution!

Review: If the roots are complex-valued...

- Suppose the two roots of the characteristic equation are r_1 and r_2
- If r_1 is complex, then r_2 will be complex (and vise versa)
- They will be *conjugate* to each other:

$$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$$

- Then using Euler's definition, we can write the general solution as

$$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t).$$

The case of Repeated Roots

Figure: This guy is stumped about repeated roots. How will he ever pass Math 307? We'd better help him out.

- Suppose the characteristic gave us the same (real) root twice
- e.g. $r_1 = r_2$
- We have one solution: $y = e^{r_1 t}$.
- But $y = Ce^{r_1 t}$ can't be the general solution (why?)
- There must be another solution out there...
- How can we find it?

A Motivating Example!

Example

Find the general solution to the ODE

$$y'' - 2y' + y = 0$$

- What's the characteristic equation?
- $r^2 - 2r + 1 = 0$
- What are the roots of the characteristic equation?
- If we factor, we get $(r-1)^2 = 0$, so the roots are 1 and 1
- So we have one solution: $y = e^t$
- Where do we go from here?

A Motivating Example!

- Great idea! Try $y = v(t)e^t$
- Then $y' = v'(t)e^t + v(t)e^t$
- and also $y'' = v''(t)e^t + 2v'(t)e^t + v(t)e^t$
- If y is a solution, then

$$\begin{aligned} 0 &= y'' - 2y' + y \\ &= v''(t)e^t + 2v'(t)e^t + v(t)e^t - 2\left(v'(t)e^t + v(t)e^t\right) + v(t)e^t \\ &= v''(t)e^t \end{aligned}$$

- Therefore $v''(t) = 0$, since e^t is never zero

A Motivating Example!

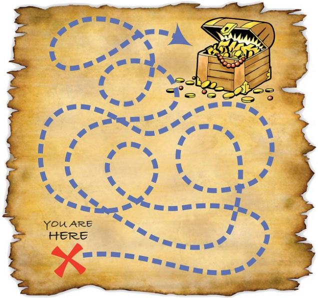
- If $v''(t) = 0$, what is v?
- $v(t) = At + B$ for some constants A, B
- We've got a new solution!

$$y = (At + B)e^t$$

- In fact, this is a two-parameter family of solutions
- It includes the old solution $y = e^t$
- As a matter of fact, it's the general solution!
- VICTORY IS OURS!

A recap of what we just did...

Figure: We can follow the steps on the right to get to the treasure. Is there possibly a more direct path?



Step 1: Figure out what the repeated root r is

Step 2: Propose a solution of the form $y = v(t)e^{rt}$

Step 3: Calculate y' and y'' and throw everything back into the ODE

Step 4: Simplify to obtain an ODE for v

Step 5: Solve for v, and write down the general solution $y = v(t)e^{rt}$

Another Example

Example

Find a solution to the initial value problem

$$y'' - y' + 0.25y = 0, \quad y(0) = 2, \; y'(0) = \frac{1}{3}$$

- The characteristic equation is

$$r^2 - r + 0.25 = 0$$

- What are the roots of this equation?
- Roots are $r_1 = r_2 = 1/2$
- So one solution is $y = e^{t/2}$

Another Example

- What do we do next?
- Propose a solution $y = v(t)e^{t/2}$.
- Then $y'(t) = v'(t)e^{t/2} + \frac{1}{2}v(t)e^{t/2}$
- and $y''(t) = v''(t)e^{t/2} + v'(t)e^{t/2} + \frac{1}{4}v(t)e^{t/2}$
- Then since y is a solution, we must have

$$\begin{aligned} 0 &= y'' - y' + 0.25y \\ &= v''(t)e^{t/2} \end{aligned}$$

- This means $v''(t) = 0$, and therefore $v(t) = At + B$
- General solution:

$$y(t) = (At + B)e^{t/2}$$

Another Example

- To solve the IVP, we need to find A and B
- Initial condition is $y(0) = 2$, $y'(0) = 1/3$
- Also $y(0) = B$
- and $y'((t) = \frac{A}{2}t + \frac{B}{2} + A)e^{t/2}$
- so $y'(0) = A + B/2$
- So we have linear system of equations

$$B = 2$$
$$A + B/2 = 1/3$$

- We get $A = -2/3$, $B = 2$ so the solution is

$$y = -\frac{2}{3}te^{t/2} + 2e^{t/2}.$$

In general, what should we expect?

- Consider the ODE

$$ay'' + by' + cy = 0$$

- and suppose that the characteristic equation

$$ar^2 + br + c = 0$$

  has a repeated root
- then the discriminant $b^2 - 4ac = 0$
- and the roots are $r_1 = r_2 = -b/2a$
- so we propose a solution of the form $y = v(t)e^{-bt/2a}$

In general, what should we expect?

- We calculate

$$y' = v'(t)e^{-bt/2a} - \frac{b}{2a}e^{-bt/2a}$$

and

$$y'' = v''(t)e^{-bt/2a} + \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$

- Then the equation

$$ay'' + by' + cy = 0$$

becomes after a bit of algebra (using the fact that $b^2 = 4ac$)

$$v''(t) = 0$$

In general, what should we expect?

- Just like we got all the other times we did repeated roots!
- Then $v(t) = At + B$
- And the general solution is therefore

$$y = Ate^{-bt/2a} + Be^{-bt/2a}$$

- Now that we've done this in the "general case", we can use it all the time!

Try it Yourself!

Find the general solutions:

- $$25y'' - 20y' + 4y = 0$$
- $$9y'' + 6y' + y = 0$$
- $$y'' - 6y' + 9y = 0$$

Try it Yourself!

Solve the initial value problem

- $$9y'' - 12y' + 4y = 0, \;\; y(0) = 2, \; y'(0) = -1$$

There's more to the story...

- The method that we first demonstrated can be applied more generally.
- For example consider the second order linear ODE

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

- Suppose we know a solution: $y = 1/t$
- How might we try to get the general solution?
- We could try a solution of the form $y(t) = v(t)/t$
- Then see if we can get a nice ODE for $v(t)$ that we can find a general solution for...
- This method is called *reduction of order*

There's more to the story...

- Let's give it a go!
- Notice that

$$y'(t) = v'(t)/t - v(t)/t^2$$

- and also

$$y''(t) = v''(t)/t - 2v'(t)/t^2 + 2v(t)/t^3$$

- Then since y is a solution to the ODE, we must have

$$\begin{aligned} 0 &= 2t^2y'' + 3ty' - y \\ &= 2tv''(t) - 4v'(t) + 4v(t)/t + 3tv'(t) - 3v(t)/t - v(t)/t \\ &= 2tv''(t) - v'(t) \end{aligned}$$

There's more to the story...

- So we've found that

$$2tv''(t) - v'(t) = 0$$

- Substituting $w'(t) = v'(t)$, we find

$$2tw' - w = 0$$

- Separable! Solution is $w(t) = At^{1/2}$.
- Then $v'(t) = Ct^{1/2}$, and therefore (for $A = \frac{2}{3}C$)

$$v(t) = At^{3/2} + B$$

- General solution: $y(t) = At^{1/2} + Bt^{-1}$

Try it Yourself!

Find the general solution

- Using the fact that $y(t) = t^2$ is a solution to the ODE

$$t^2 y'' - 4ty' + 6y = 0, \quad t > 0$$

find the general solution to the ODE

Review!

Today:

- What happens when the characteristic polynomial has repeated roots
- Reduction of order

Next time:

- 2nd-Order Nonhomogeneous Linear ODEs. with Constant Coefficients
- Method of Undetermined Coefficients