Math 307 Lecture 13

Nonhomogeneous Equations and the Method of Undetermined Parameters

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Today!

Last time:

- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

This time:

- 2nd-Order Nonhomogeneous Linear ODEs. with Constant Coefficients
- Method of Undetermined Coefficients

Next time:

- More on 2nd-Order Nonhomogeneous Linear ODEs. with Constant Coefficients

Outline

Nonhomogeneous Equations

- Recall that a general second order linear equation is something of the form

$$a(t)y'' + b(t)y' + c(t)y = f(t),$$

with a, b, c and f are functions.
- It is *nonhomogeneous* if and only if f is nonzero.
- For example the equation

$$y'' = 3y' - 4y = 3e^{2t}$$

is nonhomogeneous.

Associated Homogeneous Equation

- Suppose we have a nonhomogeneous equation ($f \neq 0$):

$$a(t)y'' + b(t)y' + c(t)y = f(t),$$

- We have the following definition.

Definition

The *associated homogeneous equation* is the equation

$$a(t)y'' + b(t)y' + c(t)y = 0,$$

- Solutions to the nonhomogeneous and homogeneous equations are intimately related.

Associated Homogeneous Equation

- In what way could they be related?

Theorem

If Y_1 and Y_2 are solutions of a nonhomogeneous lnear equation, then $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation.

- Why is this?
- It's because linear differential equations act linearly on y
- To understand what we mean by this, we need to think about linear differential equations in a new way!

Linear Equations as Linear Operators

- For any function y, we define

$$L[y] = a(t)y'' + b(t)y' + c(t)y.$$

- Notice that if y_1 and y_2 are functions, and A and B are constants then (check this!)

$$L[Ay_1 + By_2] = AL[y_1] + BL[y_2].$$

- Also the equation

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

may be written as $L[y] = f(t)$.

Linear Equations as Linear Operators

- Let Y_1 and Y_2 be solutions of

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

- Then $L[Y_1] = f(t)$ and $L[Y_2] = f(t)$
- Then

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = f(t) - f(t) = 0.$$

- Hence

$$a(t)(Y_1 - Y_2)'' + b(t)(Y_1 - Y_2)' + c(t)(Y_1 - Y_2) = 0$$

- This shows why our theorem is true

Finding general solutions

- We have the following consequence of the previous theorem

Theorem

If Y is any solution to a nonhomogeneous linear equation, and y_1 and y_2 are (independent) solutions of the corresponding homogeneous linear equation, then the general solution to the nonhomogeneous equation is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + Y(t)$$

- So how can we find the general solution to an inhomogeneous equation?
- Find the general solution to the homogeneous equation...
- and add to it any one *inhomogeneous* solution!

A First Example

Example

Find the general solution of the equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

- First we find the general solution of the corresponding homogeneous equation
$$y'' - 3y' - 4y = 0.$$
- The corresponding characteristic polynomial is $r^2 - 3r - 4$, which has roots $r_1 = 4$ and $r_2 = -1$.
- Therefore the general solution to the homogeneous equation is
$$y_h = C_1 e^{4t} + C_2 e^{-t}.$$

A First Example

- Now we need to try to find a *particular solution* $Y(t)$ to the inhomogeneous equation
- How should we go about this?
- Try to guess a reasonable form for Y. We guess $Y(t) = Ae^{2t}$ for some constant A.
- Then $Y' = 2Ae^{2t}$ and $Y'' = 4Ae^{2t}$, so that

$$Y'' - 3Y' - 4Y = 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = -6Ae^{2t}.$$

- Since $Y'' - 3Y' - 4Y = 3e^{2t}$, this means $A = -1/2$, so that $Y(t) = -\frac{1}{2}e^{2t}$
- The general solution is then

$$y = y_h + Y = C_1e^{4t} + C_2e^{-t} - \frac{1}{2}e^{2t}.$$

A Second Example

Example

Find the general solution of the equation

$$y'' - 3y' - 4y = 2\sin(t).$$

- First we find the general solution of the corresponding homogeneous equation

$$y'' - 3y' - 4y = 0.$$

- It's the same as last time! The general solution is

$$y_h = C_1 e^{4t} + C_2 e^{-t}.$$

A Second Example

- What about a particular solution $Y(t)$?
- A slick trick is to instead consider the complex equation

$$\widetilde{y}'' - 3\widetilde{y}' - 4\widetilde{y} = 2e^{it}.$$

- We've replace y with $\widetilde{y}$ to remind ourselves that its a new equation
- Why is this a good idea?
- Suppose $\widetilde{Y}$ is a particular complex solution. If we define $Y = \Im(\widetilde{Y})$ (the imaginary part of Y) then

$$\begin{aligned} Y'' - 3Y' - 4Y &= \Im\widetilde{Y}'' - 3\Im\widetilde{Y}' - 4\Im\widetilde{Y} \\ &= \Im(\widetilde{Y}'' - 3\widetilde{Y}' - 4\widetilde{Y}) \\ &= \Im(2e^{it}) = 2\sin(t) \end{aligned}$$

A Second Example

- So if we can find $\widetilde{Y}$ and take its imaginary component, we get a particular solution to the original equation!
- How can we find a particular solution $\widetilde{Y}$ to the complex equation then?
- It again seems reasonable to try $\widetilde{Y} = Ae^{it}$ for some undetermined constant A
- Then $\widetilde{Y}' = iAe^{it}$ and $\widetilde{Y}'' = -Ae^{it}$, so that

$$\widetilde{Y}'' - 3\widetilde{Y}' - 4\widetilde{Y} = -Ae^{2t} - 3iAe^{2t} - 4Ae^{2t} = (-5 - 3i)Ae^{2t}.$$

- Then since $\widetilde{Y}'' - 3\widetilde{Y}' - 4\widetilde{Y} = 2e^{-it}$, we must have $(-5 - 3i)A = 2$

A Second Example

- Dividing both sides by $(-5-3i)$ we obtain

$$A = \frac{2}{-5-3i} = \frac{2}{-5-3i}\frac{-5+3i}{-5+3i} = \frac{-10+6i}{34} = \frac{-5}{17} + \frac{3}{17}i$$

- Putting this into our expression for $\widetilde{Y}$, we get

$$\widetilde{Y} = \left(\frac{-5}{17} + \frac{3}{17}i\right)e^{it} = \left(\frac{-5}{17} + \frac{3}{17}i\right)(\cos(t) + i\sin(t))$$
$$= \left(-\frac{5}{17} + \frac{3}{17}i\right)\cos(t) + \left(-\frac{3}{17} - \frac{5}{17}i\right)\sin(t)$$

A Second Example

- Our particular solution Y can then be found by taking the imaginary component of $\widetilde{Y}$
- Therefore we have our particular solution!

$$Y = \Im(\widetilde{Y}) = \frac{3}{17}\cos(t) - \frac{5}{17}\sin(t)$$

- General solution to the inhomogeneous equation is then

$$y = y_h + Y = C_1 e^{4t} + C_2 e^{-t} + \frac{3}{17}\cos(t) - \frac{5}{17}\sin(t)$$

A Third Example

Example

Find the general solution of the equation

$$y'' - 3y' - 4y = 2\cos(t).$$

- First we find the general solution of the corresponding homogeneous equation

$$y'' - 3y' - 4y = 0.$$

- It's the same as last time! The general solution is

$$y_h = C_1 e^{4t} + C_2 e^{-t}.$$

A Third Example

- What about a particular solution $Y(t)$?
- A slick trick is to instead consider the complex equation

$$\widetilde{y}'' - 3\widetilde{y}' - 4\widetilde{y} = 2e^{it}.$$

- We've replace y with $\widetilde{y}$ to remind ourselves that its a new equation
- Why is this a good idea?
- Suppose $\widetilde{Y}$ is a particular complex solution. If we define $Y = \Re(\widetilde{Y})$ (the real part of Y) then

$$\begin{aligned} Y'' - 3Y' - 4Y &= \Re\widetilde{Y}'' - 3\Re\widetilde{Y}' - 4\Re\widetilde{Y} \\ &= \Re(\widetilde{Y}'' - 3\widetilde{Y}' - 4\widetilde{Y}) \\ &= \Re(2e^{it}) = 2\cos(t) \end{aligned}$$

A Third Example

- So if we can find $\widetilde{Y}$ and take its real component, we get a particular solution to the original equation!
- How can we find a particular solution $\widetilde{Y}$ to the complex equation then?
- We did this already earlier! We found

$$\widetilde{Y} = \left(-\frac{5}{17} + \frac{3}{17}i\right)\cos(t) + \left(-\frac{3}{17} - \frac{5}{17}i\right)\sin(t)$$

- And therefore we have our particular solution!

$$Y = \Re(\widetilde{Y}) = -\frac{5}{17}\cos(t) - \frac{3}{17}\sin(t)$$

- So the general solution is

$$y = y_h + Y = C_1 e^{4t} + C_2 e^{-t} - \frac{5}{17}\cos(t) - \frac{3}{17}\sin(t)$$

A Fourth Example

Example

Find the general solution of the equation

$$y'' - 3y' - 4y = 3\cos(t) - 7\sin(t).$$

- First we find the general solution of the corresponding homogeneous equation

$$y'' - 3y' - 4y = 0.$$

- It's the same as last time! The general solution is

$$y_h = C_1 e^{4t} + C_2 e^{-t}.$$

A Fourth Example

- What about a particular solution?
- Let $L[y] = y'' - 3y' - 4y$
- Earlier, we found functions Y_1 and Y_2 satisfying $L[Y_1] = 2\sin(t)$ and $L[Y_2] = 2\cos(t)$, namely

$$Y_1 = \frac{3}{17}\cos(t) - \frac{5}{17}\sin(t)$$

$$Y_2 = -\frac{5}{17}\cos(t) - \frac{3}{17}\sin(t)$$

A Fourth Example

- Therefore, if we take $Y = \frac{-7}{2}Y_1 + \frac{3}{2}Y_2$, then

$$L[Y] = L\left[\frac{-7}{2}Y_1 + \frac{3}{2}Y_2\right] = \frac{-7}{2}L[Y_1] + \frac{3}{2}L[Y_2]$$
$$= \frac{-7}{2}(2\sin(t)) + \frac{3}{2}(2\cos(t)) = -7\sin(t) + 3\cos(t).$$

- This Y is a particular solution!

$$Y = \frac{-18}{17}\cos(t) + \frac{13}{17}\sin(t)$$

- General solution is then

$$y = y_h + Y = C_1e^{4t} + C_2e^{-t} - \frac{18}{17}\cos(t) + \frac{13}{17}\sin(t)$$

A Fifth Example

Example

Find the general solution of the equation

$$y'' - 3y' - 4y = -8e^t\cos(2t).$$

- First we find the general solution of the corresponding homogeneous equation

$$y'' - 3y' - 4y = 0.$$

- It's the same as last time! The general solution is

$$y_h = C_1e^{4t} + C_2e^{-t}.$$

A Fifth Example

- What about a particular solution?
- A slick trick is to consider the complex equation

$$\widetilde{y}'' - 3\widetilde{y}' - 4\widetilde{y} = -8e^{(1+2i)t}.$$

- We've replace y with $\widetilde{y}$ to remind ourselves that its a new equation
- Why is this a good idea?
- Suppose $\widetilde{Y}$ is a particular complex solution. If we define $Y = \Re(\widetilde{Y})$ (the real part of Y) then

$$\begin{aligned} Y'' - 3Y' - 4Y &= \Re\widetilde{Y}'' - 3\Re\widetilde{Y}' - 4\Re\widetilde{Y} \\ &= \Re(\widetilde{Y}'' - 3\widetilde{Y}' - 4\widetilde{Y}) \\ &= \Re(-8e^{(1+2i)t}) = -8e^{t}\cos(2t) \end{aligned}$$

A Fifth Example

- So if we can find $\widetilde{Y}$ and take its real component, we get a particular solution to the original equation!
- How can we find a particular solution $\widetilde{Y}$ to the complex equation then?
- It again seems reasonable to try $\widetilde{Y} = Ae^{(1+2i)t}$ for some undetermined constant A
- Then $\widetilde{Y}' = (1+2i)Ae^{it}$ and $\widetilde{Y}'' = (-3+4i)Ae^{it}$, so that

$$\begin{aligned}\widetilde{Y}'' - 3\widetilde{Y}' - 4\widetilde{Y} &= (-3+4i)Ae^{2t} - 3(1+2i)Ae^{2t} - 4Ae^{2t}\\ &= (-10-2i)Ae^{2t}\end{aligned}$$

- Then since $\widetilde{Y}'' - 3\widetilde{Y}' - 4\widetilde{Y} = -8e^{-(1+2i)t}$, we must have $(-10-2i)A = -8$

A Fifth Example

- Dividing both sides by $(-8 - 2i)$ we obtain

$$A = \frac{-8}{-10 - 2i} = \frac{4}{5 + i} = \frac{4}{5 + i}\frac{5 - i}{5 - i} = \frac{20 - 4i}{26} = \frac{10}{13} - \frac{2}{13}i$$

- Putting this into our expression for $\widetilde{Y}$, we get

$$\widetilde{Y} = \left(\frac{10}{13} - \frac{2}{13}i\right) e^{(1+2i)t} = \left(\frac{10}{13} - \frac{2}{13}i\right) e^{t}(\cos(2t) + i\sin(2t))$$
$$= \left(\frac{10}{13} - \frac{2}{13}i\right) e^{t}\cos(2t) + \left(\frac{2}{13} + \frac{10}{13}i\right) e^{t}\sin(2t)$$

A Fifth Example

- Our particular solution Y can then be found by taking the real component of $\widetilde{Y}$
- And therefore we have our particular solution!

$$Y = \Re(\widetilde{Y}) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(2t)$$

- General solution to the inhomogeneous equation is then

$$y = y_h + Y = C_1 e^{4t} + C_2 e^{-t} + \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(2t)$$

Try It Yourself!

Find the general solutions of the following equations:

- $y'' - 2y' - 3y = 3e^{2t}$
- $y'' - 2y' - 3y = e^{-t}\sin(t)$
- $y'' - 2y' - 3y = e^{-t}\cos(t)$
- $y'' - 2y' - 3y = 2e^{2t} - 3e^{-t}\cos(t) + 4e^{-t}\sin(t)$