Math 307 Lecture 15

Mechanical and Electrical Vibrations

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Today!

Last time:

- Nonhomogeneous Equations and the Method of Undetermined Coefficients

This time:

- Mechanical and Electrical Vibrations

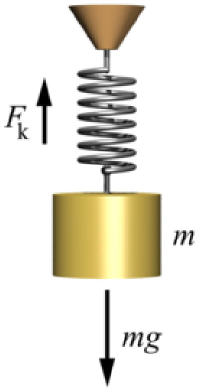
Next time:

- More on Mechanical and Electrical Vibrations

Outline

An Ideal Mass-Spring System

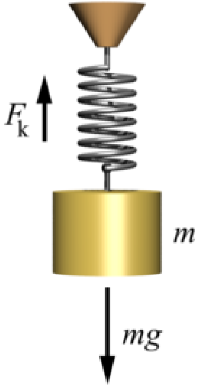
Figure: A Mass Spring System



- A spring has natural resting length ℓ
- Attach a mass, stretches to length L
- Forces then balanced: $k(L-\ell) = mg$
- We call L the *resting length* of the mass spring system.
- Now stretch the spring by an additional amount u
- Then length is $L + u$.

An Ideal Mass-Spring System

Figure: A Mass Spring System

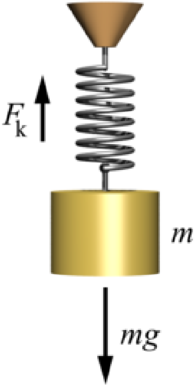


- The forces on the spring are then $F = F_{\text{grav}} + F_{\text{spring}}$
- $F_{\text{grav}} = mg$
- $F_{\text{spring}} = -k(L + u - \ell)$
- $F = mg - k(L + u - \ell)$
- Since $k(L - \ell) = mg$, this means $F = -ku$.
- From Newton's law $F = mu''$. Therefore u satisfies the equation

$$mu'' = -ku.$$

An Ideal Mass-Spring System

Figure: A Mass Spring System



- This is a second order linear homogeneous ODE with constant coefficents
- The corresponding characteristic equation has complex roots $\pm i\sqrt{\frac{k}{m}}$.
- The general solution is therefore of the form

$$u = A\cos(\omega t) + B\sin(\omega t),$$

for $\omega = \sqrt{k/m}$.

An Example

Example

Suppose a mass weighing 4 lbs stretches a spring 2 inches. Then suppose the resulting mass spring system is stretched an additional six inches from its resting position, and then released. Determine the position u of the mass relative to its resting position as a function of time.

- We know that the mass m satisfies $mg = 4\text{lbs}$. Since $g \approx 32$ ft/s^2, it follows that $m = \frac{1}{8}$ lb$\cdot$s$^2/$ft.
- Also know $L - \ell = 2$ inches $= 1/6$ ft.
- Need to find the spring constant. How can we?
- Balance of forces! $k(L - \ell) = mg$. So $k = \frac{4\text{lbs}}{1/6\text{ft}} = 24\text{lbs/ft}$.

An Example

- Therefore u satisfies the equation

$$\frac{1}{8}u'' + 24u = 0$$

- The corresponding characteristic equation has complex roots $\pm i\sqrt{192}$, so the general solution is

$$u = A\cos(\sqrt{192}t) + B\sin(\sqrt{192}t).$$

- What is our initial condition?
- $u(0) = 6$ in $= 1/2$ ft. and $u'(0) = 0$ (released from rest)
- Therefore $A = 1/2$ and $B = 0$, so that $u = \cos(\sqrt{192}t)$.

Some Trigonometry

- Often times in the next part, we'll be dealing with equations of the form
$$u = A\cos(\omega t) + B\sin(\omega t).$$
- It will be important for us to note that for some choice of δ and R, this may be rewritten as
$$u = R\cos(\omega t - \delta)$$
- In this form, δ is called the *phase* or *phase angle*.
- Specifically $R = \sqrt{A^2 + B^2}$
- and also $\delta = \tan^{-1}(B/A)$

Some Vocabulary

- If $u = R\cos(\omega t - \delta)$ is a solution to the spring equation

$$mu'' + ku = 0,$$

- We call ω the *natural frequency*
- We call $2\pi/\omega$ the *period*
- We call R the *amplitude*
- We call δ the *phase*

A Damped Mass-Spring System

Figure: Friction causes realistic mass-spring systems to be damped

- Real-world mass-spring systems are not ideal
- Real springs aren't ideal (can be rusty, noisy, etc) so oscillating mass slows down over time
- This adds an additional *drag term* to the force proportional to velocity u'
- $F_{\text{drag}} = -\gamma u'$
- Here γ is a constant, called the *damping constant*

A Damped Mass-Spring System

- The equation of motion for an ideal spring is

$$mu'' + ku = 0.$$

- The equation of motion for a damped spring is

$$mu'' + \gamma u' + ku = 0.$$

- m, k and γ are all nonnegative constants
- When $\gamma = 0$, the motion is ideal

An Example

Example

Suppose that a mass spring system has spring constant $k = 1$ lbs/ ft, mass $m = 1$ lb·s^2/ft and damping constant $\gamma = 0.125$ lb·s/ft. If initially the spring is stretched 2 feet and then released, determine the position u of the mass relative to its resting position as a function of time.

- u is determined by the initial value problem

$$u'' + 0.125u' + u = 0, \quad u(0) = 2, \; u'(0) = 0$$

- The roots of the characteristic polynomial are $\frac{-1}{16} \pm \frac{\sqrt{255}}{16}$.

An Example

- This means that the general solution is

$$u = e^{-t/16}\left[A\cos\left(\frac{\sqrt{255}}{16}t\right) + B\sin\left(\frac{\sqrt{255}}{16}t\right)\right].$$

- To satisfy the initial conditions, we must choose $A = 2$ and $B = 2/\sqrt{255}$
- Then the solution of the initial value problem is

$$u = e^{-t/16}\left[2\cos\left(\frac{\sqrt{255}}{16}t\right) + \frac{2}{\sqrt{255}}\sin\left(\frac{\sqrt{255}}{16}t\right)\right].$$

An Example

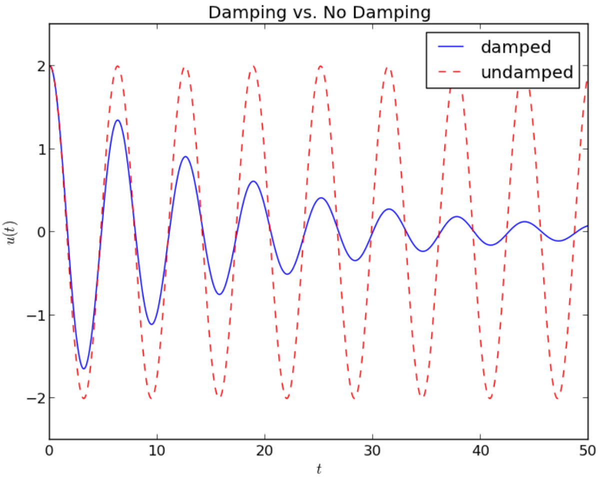
- Using our trig stuff with $R = \sqrt{A^2 + B^2}$ and $\delta = \tan^{-1}(B/A)$, we find

$$u = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta\right)$$

- where $\delta = \tan^{-1} \frac{1}{\sqrt{255}} \approx 0.0625$
- Compare this solution to the solution we'd get without damping ($\gamma = 0$). The undamped solution would have been

$$u = 2\cos(t).$$

An Example


Damping vs. No Damping
damped
undamped
2
1
0
−1
−2
$u(t)$
0
10
20
30
40
50
t

Some Vocabulary

- Unless the system is *overdamped* (meaning $\gamma \geq 2\sqrt{km}$, ie. lots of damping), the solution of the damped spring equation

$$mu'' + \gamma u' + ku = 0,$$

will be

$$Re^{-\gamma t/2m}\cos(\mu t - \delta)$$

for some constants R, μ and δ.

- We call μ the *quasi-frequency* (for small γ, close to undamped frequency)
- We call $2\pi/\mu$ the *quasi-period* (equal to the time between successive maxima or successive minima)

Example Problem!

Example

A mass weighing 2 lb stretches a spring 6 inches. If the mass is pulled down an additional 2 inches and then released, and if there is no damping, determine the position u of the mass at any time t. Find the frequency, period, and amplitude of the motion.

Example Problem!

Example

A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm$/$s, and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?

Example Problem!

Example

A spring is stretched 10 cm by a force of 3 Newtons. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 Newtons when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position u at any time t. Find the quasifrequency μ and the ration of μ to the natural frequency of the corresponding undamped motion.

Review!

Today:

- Mass-spring systems

Next time:

- More mechanical and electrical vibrations