Math 307 Lecture 17

Forced Vibrations

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Today!

Last time:

- More on Mechanical and Electrical Vibrations

This time:

- Forced vibrations

Next time:

- Review for final exam

Outline

What is Forced Vibration?

- Forced vibration is what occurs when we have an equation of the form
$$ay'' + by' + cy = R\cos(\omega t).$$
- where a, b, c are positive constants
- Intuitively, we should think about a spring with an external periodic force applied.
- What solutions look like depend very strongly on the values of a, b, c, R, and ω
- Today we will look at behavior when $b > 0$
- Later we will study separately the case when $b = 0$

Examples of Forced Vibration

- Forced vibrations occur naturally all the time!
- For example one could think of
  - A mass-spring system, with an external force applied
  - A LCR-circuit, where the voltage input $E(t)$ is not constant
- What does forced vibration look like in comparison to unforced vibration?
- How does forced vibration behave at large timescales?

LCR-circuit example

- Last class, we found that the current in an LCR-circuit with inductance L, capacitance C, and resistance R (all positive constants) satisfies the second-order equation

$$LI'' + RI' + I/C = E'(t)$$

- where here $E(t)$ is the voltage input into the LCR circuit by some power source
- power source could be a battery, or similar device
- could also be power from an electrical outlet or a motor with or without an alternator

LCR-circuit example

- Suppose that our voltage source is V_0 volts
- Depending on the kind of source, this may mean two different things
- If the source is a battery, then the voltage is *direct* and $E(t) = V_0$, so that $E'(t) = 0$
- If the source is a wall outlet or alternator, the voltage is *alternating*
- In this case V_0 typically refers to the *rms voltage*, so that the true voltage input is $E(t) = \sqrt{2}V_0 \sin(\omega t)$, where ω is the electrical angular frequency of the power source
- For wall outlets in the U.S.A., $\omega/2\pi = 60$ Hz

LCR-circuit: direct current case

- Suppose that our voltage source is V_0 volts in direct current (ie. from a battery or DC generator)
- Then $E(t)$ is constant, so $E'(t) = 0$
- Also assume that $R > 0$ (which is *always* true in real life)
- The current satisfies the equation

$$LI'' + RI' + I/C = 0$$

- and for $R^2 < 4L/C$, the solution is of the form

$$I = re^{-Rt/2L}\cos(\mu t - \delta)$$

- where $\mu = (\sqrt{R^2 - 4L/C})/2L$, and r, δ are constants depending on the initial conditions $I(0)$ and $I'(0)$

LCR-circuit: direct current case

- How does $I(t)$ behave at large time?
- It becomes very small, approaching zero exponentially fast
- Comparatively, the behavior of the current in the alternating voltage source case is much more interesting

LCR-circuit: alternating current case

- Suppose that our voltage source is V_0 volts in alternating current (ie. from a wall socket or alternator)
- Then $E(t) = \sqrt{2}V_0 \sin(\omega t)$, so $E'(t) = \sqrt{2}\omega V_0 \cos(\omega t)$
- Also assume that $R > 0$ (which is *always* true in real life)
- The current satisfies the equation

$$LI'' + RI' + I/C = \sqrt{2}\omega V_0 \cos(\omega t)$$

- In this case, the solution has two components

$$I = \overbrace{c_1 I_1 + c_2 I_2}^{\text{solution to homogeneous equation}} + \overbrace{I_p}^{\text{particular solution}}$$

- Where c_1, c_2 are constants, depending on initial condition

LCR-circuit: alternating current case

- As the solution to the homogeneous equation we derived before shows us, the solution to the homogeneous equation will become very small at large times
- Therefore, at large times, the general solution will look very similar to the particular solution
- For this reason, we often call $c_1 I_1 + c_2 I_2$ the *transient solution* (in the case of circuits, also called a transient current
- And we call I_p the *steady state solution* or *forced response* (in the circuit case, also called *steady state current*).

LCR-circuit: alternating current case

- Thus we write

$$I = I_{\text{transient}} + I_{\text{steady}},$$

- where I_{steady} is a particular solution of

$$LI'' + RI' + I/C = \sqrt{2}\omega V_0 \cos(\omega t)$$

- and where $I_{\text{transient}}$ is a solution of the corresponding homogeneous equation

$$LI'' + RI' + I/C = 0$$

- which we know get exponentially smaller as time increases when the resistance R is not zero

LCR-circuit: alternating current case

- Straightforward but tedious algebraic calculation may be used to show

$$I_{\text{steady}} = r\cos(\omega t - \delta),$$

- where

$$r = \frac{\sqrt{2}\omega V_0}{\Delta}, \quad \cos(\delta) = \frac{L(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin(\delta) = \frac{R\omega}{\Delta}$$

- and also

$$\Delta = \sqrt{L^2(\omega_0^2 - \omega^2)^2 + R^2\omega^2}, \quad \omega_0 = 1/\sqrt{LC}$$

- so as time goes on, the current behaves like a wave with frequency the same as the frequency of $E'(t)$

LCR-circuit example

Example

Suppose that an LCR-circuit has a 1 H inductor, a 1 F capacitor, and a 0.125Ω resistor. Also suppose that the initially $I'(0) = 0$ and $I(0) = 2$. If a voltage source of 0.25 volts is connected, determine the current as a function of time.

We consider two cases:

- when the voltage source is direct current (DC)
- when the voltage source is alternating current (AC)

LCR-circuit example: DC case

- Assume that the voltage source is direct
- We have that $C = 1$, $L = 1$, and $R = 0.125$.
- In this case, $E(t) = 0.25$, $E'(0) = 0$, so that I is a solution to the initial value problem

$$I'' + 0.125I' + I = 0, \ \ I(0) = 2, \ \ I'(0) = 0$$

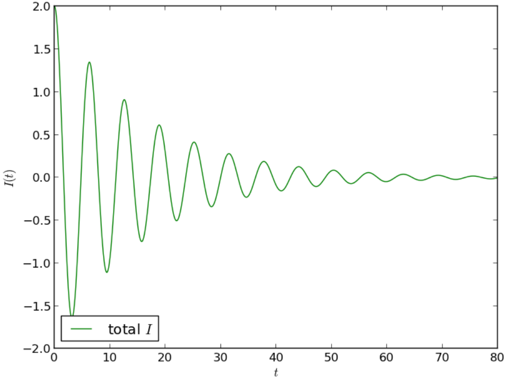
- We solved this last time, and got

$$I = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta\right)$$

for $\delta = \tan^{-1} \frac{1}{\sqrt{255}} \approx 0.0625$

LCR-circuit example: DC case

Figure: Current in the LCR circuit $L = 1$, $C = 1$, $R = 0.125$, $E = 0.25$, with the initial condition $I(0) = 2$ and $I'(0) = 0$



LCR-circuit example: AC case

- Assume the voltage source is alternating with angular frequency $\omega = 1$
- We have that $C = 1$, $L = 1$, and $R = 0.125$.
- In this case, it's the RMS voltage that is 0.25 volts
- This means that $E(t) = 0.25\sqrt{2}\sin(t)$, and $E'(0) = 0.25\sqrt{2}\cos(t)$, so that I is a solution to the initial value problem

$$I'' + 0.125I' + I = 0.25\sqrt{2}\cos(t)$$

LCR-circuit example: AC case

- As per our previous discussion, we write

$$I = I_{\text{transient}} + I_{\text{stable}},$$

- where $I_{\text{transient}}$ is a solution of the homogeneous equation

$$I'' + 0.125I' + I = 0$$

- and where I_{stable} is a particular solution of

$$I'' + 0.125I' + I = 0.25\sqrt{2}\cos(t)$$

LCR-circuit example: AC case

- Using the equation of the previous slide,

$$I_{\text{steady}} = r\cos(\omega t - \delta),$$

- where $\omega_0 = 1$, $\Delta = \sqrt{0.125} = \sqrt{2}/4$, $\delta = \pi/2$, and $r = 1$, so that

$$I_{\text{steady}} = \cos\left(t - \frac{\pi}{2}\right)$$

LCR-circuit example: AC case

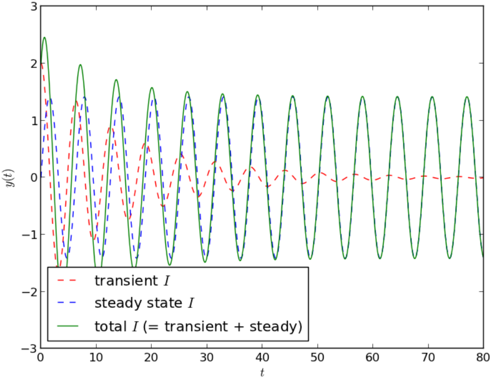
- Also, since $I_{\text{transient}}$ is a solution of the homogeneous equation, we know that

$$I_{\text{transient}} = r_0 e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta_0\right),$$

- where r_0 and δ_0 are some constants depending on our initial condition $I(0) = 2$ and $I'(0) = 0$.
- EXERCISE: determine r_0 and δ_0

LCR-circuit example: AC case

Figure: Current in the LCR circuit $L = 1$, $C = 1$, $R = 0.125$, $E = 0.25$, with the initial condition $I(0) = 2$ and $I'(0) = 0$



Ideal Spring Example

- this time, we look at the case when $b = 0$

Example

Suppose that an ideal spring system is being acted on by an external force so as to satisfy the equation

$$mu'' + ku = R\cos(\omega t)$$

if initially $u(0) = u'(0) = 0$, what is $u(t)$?

- what the solution to this equation looks like depends on the *frequency of forcing* ω
- in particular it depends on whether or not ω is equal to the *resonant frequency* of the mass spring system $\omega_0 = \sqrt{k/m}$

Ideal Spring Example: when $\omega \neq \omega_0$

- Assume $\omega \neq \omega_0$. Then the general solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{R}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

- applying our initial condition, $c_1 = -\frac{R}{m(\omega_0^2 - \omega^2)}$ and $c_2 = 0$.
- Solution is therefore

$$u(t) = \frac{2R}{m(\omega_0^2 - \omega^2)} (\cos(\omega t) - \cos(\omega_0 t))$$

Ideal Spring Example: when $\omega \neq \omega_0$

- Notice that by some trig identities

$$\cos(\omega t) - \cos(\omega_0 t) = \sin\left(\frac{\omega - \omega_0}{2} t\right) \sin\left(\frac{\omega + \omega_0}{2} t\right)$$

- so that our solution may be written in the form

$$u(t) = \frac{2R}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega - \omega_0}{2} t\right) \sin\left(\frac{\omega + \omega_0}{2} t\right)$$

Ideal Spring Example: when $\omega \neq \omega_0$

- Thus in this case, the solution y is a sine wave with amplitude changing over time

$$u(t) = A(t) \sin\left(\frac{\omega + \omega_0}{2} t\right)$$

- where $A(t) = \frac{2R}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega - \omega_0}{2} t\right)$ is the amplitude of the sine wave
- In sound waves, this motion is called a *beat*
- In electronics, this motion is called *amplitude modulation*

Ideal Spring Example: when $\omega = \omega_0$

- Assume $\omega = \omega_0$. This situation is called *resonance*
- In this case, the forcing function on the right hand side is also a solution of the homogeneous equation.
- Then the general solution is therefore

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{R}{2m\omega_0} t \sin(\omega_0 t)$$

- The initial condition implies $c_1 = 0$ and $c_2 = 0$, so that the solution is

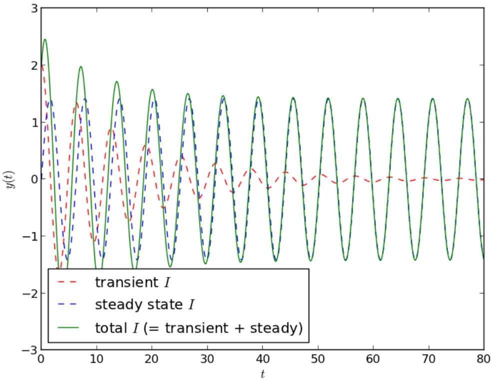
$$u(t) = \frac{R}{2m\omega_0} t \sin(\omega_0 t)$$

Ideal Spring Example: when $\omega = \omega_0$

- As time goes on, the amplitude gets very large (grows linearly)!
- This is not physical ... as soon as the amplitude gets too big, the equation is no longer a good model for the physics of the situation.
- However, it corresponds to the fact that when the resistance is very small, and forcing frequency is close to the resonant frequency, then the amplitude of the resulting oscillation may be quite large
- http://www.youtube.com/watch?v=JDnNmLkQ3Bc

Ideal Spring Example: when $\omega \neq \omega_0$

Figure: Forced spring oscillation without damping, for $\omega = 1.0$, $\omega_0 = 1.0$, $R = 0.125$, and $m = 1$



Review!

Today:

- Forced vibrations

Next time:

- Review for second midterm