Math 307 Lecture 2

First Order Linear Equations!

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January 3, 2016

Today!

Last time:

- What is a differential equation?
- First order differential equations
- Separable and homogeneous equations

This time:

- First-Order Linear Equations
- Exact Linear Equations
- Solving Exact Linear Equations

Next time:

- More First-Order Linear Equations
- Method: Integrating Factors
- Method: Variation of Parameters

Outline

What is a First-Order Linear Equation?

Recall that a first order ordinary differential equation (ODE) is an equation of the form

$$\frac{dy}{dt} = f(t, y).$$

Question

When is a first order differential equation *linear*?

Definition

A first order ODE is called *linear* if it can be written in the form

$$y' = p(t)y + q(t)$$

for some functions $p(t), q(t)$

Examples of Linear Equations

- Linear: $y' = 2ty + 3\sin(t)$
- Not linear: $y' = y^2$
- Linear: $y' = \cos(3t)y + e^t$
- Not linear: $yy' = t^2$
- Linear: $ty' + t^2y = \sin(t)$ (divide by t on both sides)

Why do we care about First-Order Linear Equations?

Why should we care about first-order linear ODE?

- Arises naturally in many situations
  - Newtons law of cooling
  - Compound interest
  - Mixing fluids in a tank
  - Velocity of a falling body with air friction
- We know how to solve them!
  - General solution for linear: involves one arbitrary constant
  - Not true for nonlinear equations (see for example $y' = y^2$)
- Can approximate solutions of nonlinear equations by solutions of linear ones
  - For example, consider the IVP $\frac{dy}{dt} = e^{ty}$; $y(0) = 0$
  - By Taylor series $e^{ty} \approx 1 + ty$
  - Solutions to IVP $\frac{dy}{dt} = 1 + ty$; $y(0) = 0$ are approximations

A First Example

Example

Let a, b be *constants*. Find a general solution to the ODE

$$\frac{dy}{dt} = ay + b$$

- We already know how to solve this! Why?
- That's right, it's separable!

$$\frac{1}{ay+b}\frac{dy}{dt} = 1$$
$$\frac{1}{ay+b}dy = dt$$

A First Example ~ Continued

Continuing our calculations...

$$\begin{aligned}
\int \frac{1}{ay+b} dy &= \int dt \\
\frac{1}{a} \ln|ay+b| &= t + C_0 \\
\ln|ay+b| &= at + C_1 & C_1 &= aC_0 \\
ay+b &= e^{at+C_1} \\
ay+b &= C_2 e^{at} & C_2 &= e^{C_1} \\
y &= C_3 e^{at} - \frac{b}{a} & C_3 &= C_2/a
\end{aligned}$$

We might in the future drop the indexing of the constants, and just let arbitrary constants be arbitrary :)

Solving Arbitrary First-Order linear ODEs

- In general, solving first-order linear equations won't be as easy :(
- Even so, we will always be able to solve them (up to an integral)
- We will give two methods for this next lecture:
  - Method of Integrating Factors
  - Method of Variation of Parameters
- Their names come from more general methods
- You should be careful to know how to solve an equation both ways!
- TODAY: how to solve *exact* linear equations

Motivating Example

Example

Find the general solution of the first-order linear ODE

$$\sin(t)y' + \cos(t)y = \sec(t)\tan(t)$$

- How might we solve this?
- Observe: $\sin(t)y' + \cos(t)y = (\sin(t)y)'$
- This means: $(\sin(t)y)' = \sec(t)\tan(t)$
- Integrating: $\sin(t)y = \sec(t) + C$

Final answer:

$$y = \frac{\sec(t) + C}{\sin(t)}$$

Exact Equations

That was cool, right?

- One question: What just happened ?!?!
- Answer: the differential equation was *exact*

What's that suppose to mean?

- Consider a first-order ODE $a(t)y' + b(t)y = c(t)$
- Note that this is linear, since it may be rewritten in the form

$$y' = (-b(t)/a(t))y + c(t)/a(t).$$

Definition

A first order linear ODE $a(t)y' + b(t)y = c(t)$ is exact if

$$a'(t) = b(t).$$

Exact Equation Practice!

Let's try to tell if the following equations are exact

- $(t+1)e^t y + te^t y' = 1$
- Answer: yes!
- $(t+1)y + ty' = e^{-t}$
- Answer: no! (Note: compare to previous)
- $\cos(t)y' = \sin(t)y + \frac{1}{1+t^2}$
- Answer: yup!
- Solutions to exact equations are easy to find...
- Today, we'll only talk about how to solve exact linear equations

Solving Exact Linear Equations

Consider a first order linear equation of the form

$$a(x)y' + b(x)y = c(x).$$

- This equation is exact if $a'(x) = b(x)$
- In this case

$$(a(x)y)' = a'(x)y + a(x)y' = b(x)y + a(x)y' = c(x)$$

- So therefore

$$(a(x)y)' = c(x)$$

Solving Exact Linear Equations

If we now integrate both sides with respect to x:

$$\int (a(x)y)' dx = \int c(x) dx.$$

$$a(x)y = \int c(x) dx.$$

we find the solution is

$$y = \frac{1}{a(x)} \int c(x) dx.$$

Another Example

Example

Find the solution to the equation

$$e^x y' + e^x y = \cos(x).$$

- This equation is exact
- In particular, $(e^x y)' = e^x y' + e^x y$
- Original equation becomes: $(e^x y)' = \cos(x)$
- Integrating: $e^x y = \sin(x) + C$
- Solution: $y = e^{-x}\sin(x) + Ce^{-x}$

Summary!

What we did today:

- We learned about linear equations
- We learned about exact equations
- We learned how to solve exact linear equations

Plan for next time:

- More on solving first order linear equations