Math 307 Lecture 2

First Order Linear Equations!

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Today!

Last time:

- First-Order Linear Equations
- Exact Equations
- Solving Exact Linear Equations

This time:

- Methods of solving First-Order Linear Equations
- Method: Integrating Factors
- Method: Variation of Parameters

Next time:

- More First-Order Linear Equations

As always, todays's class will in NO WAY involve squirrels.

NO SQUIRRELS

NONE

Outline

Motivating Example

Example

Find the general solution of the first-order linear ODE

$$y' + \cot(t)y = \sec^2(t)$$

- Is it exact?
- Nope, so how do we solve this?
- Multiply both sides by $\sin(t)$:

$$\sin(t)y' + \cos(t)y = \sec(t)\tan(t)$$

- Now it's exact, so we know how to solve it!
- Takeaway: we made the ODE exact by multiplying both sides of the equation by a "nice" function

Integrating Factors

Consider a first order linear ODE

$$a(t)y' + b(t)y = c(t)$$

Definition

A function $\mu(t)$ is a *integrating factor* for this equation if the equation

$$a(t)\mu(t) + b(t)\mu(t)y' = c(t)\mu(t)$$

is an exact equation.

- In other words, an integrating factor is a function that turns ODE's exact when you multiply by it!
- This integrating factor is the "nice" function from the previous example

Integrating Factors: The Linear Case

Let's think about a general first-order linear ODE

$$y' = p(t)y + q(t)$$

- Is this equation exact, do you think?
- Not usually: only if p is constant

Question

Can we find an integrating factor for this equation, making it exact?

- Yes we can! How do we do it?

Integrating Factors: The Linear Case $\sim$ Continued

- Assume that an integrating factor $\mu(t)$ exists
- Then this must be exact:

$$\mu(t)y' = p(t)\mu(t)y + q(t)\mu(t)$$

- Implying that $\mu'(t) = -p(t)\mu(t)$; a first-order separable ODE!
- We can solve it to get an integrating factor
- $\mu(t) = e^{-\int p(t)dt}$

Squirrel says:

That’s NUTS! How about an example?

Method of Integrating Factors: Example

Example

Use the method of integrating factors to find the general solution of the first-order linear ODE

$$y' = 2y + te^{3t}.$$

- Check: is it exact? (Nope!)
- So we need an integrating factor $\mu(t)$
- Note that in the previous notation $p(t) = 2$
- Thus from before, $\mu(t) = e^{-\int 2dt} = e^{-2t}$

Method of Integrating Factors: Example $\sim$ Continued

Now our equation is the exact linear equation

$$e^{-2t}y' - 2e^{-2t}y = te^{t}.$$

- Notice that $(e^{-2t}y)' = e^{-2t}y' - 2e^{-2t}$ and therefore

$$\begin{aligned} (e^{-2t}y)' &= te^{t} \\ \int (e^{-2t}y)' dt &= \int te^{t} dt \\ e^{-2t}y &= te^{t} - e^{t} + C \\ y &= te^{3t} - e^{3t} + Ce^{2t} \end{aligned}$$

Summary: Method of Integrating Factors

To solve the equation

$$y' = p(t)y + q(t).$$

- Multiply both sides by $\mu(t) = e^{-\int p(t)dt}$ to get exact equation
$$\mu(t)y' = p(t)\mu(t)y + q(t)\mu(t)$$
- Group y-terms:
$$(\mu(t)y)' = q(t)\mu(t)$$
- Integrate and solve for y:
$$y = \frac{1}{\mu(t)} \int q(t)\mu(t)dt$$

Homogeneous Equation

Consider the first order linear ODE

$$y' = p(x)y + q(x)$$

Definition

The *homogeneous equation* associated to a first-order linear ODE is

$$y' = p(t)y$$

- CAUTION! The associated homogeneous equation isn't a homogeneous equation in the sense of last time
- Notice that this equation is *separable* (so we can solve it using the methods of last time!)

The Method

Let y_h be a solution of the homogeneous equation associated to the linear ODE

$$y' = p(t)y + q(t)$$

- Define $v(t)$ implicitly by $y = vy_h$. Then

$$\begin{aligned}
y' &= p(t)y + q(t)\\
v'y_h + vy_h' &= p(t)y + q(t)\\
v'y_h + vp(t)y_h &= p(t)y + q(t)\\
v'y_h + p(t)y &= p(t)y + q(t)\\
v'y_h &= q(t)\\
v &= \int \frac{q(t)}{y_h}dt \quad \implies \quad y = y_h \int \frac{q(t)}{y_h}dt
\end{aligned}$$

Squirrel says:

Aaak! Choking on abstract nonsense!!

Method of Variation of Parameters: Example

Example

Use the method of integrating factors to find the general solution of the first-order linear ODE

$$y' = 2y + t^2 e^{2t}$$

- The corresponding homogeneous equation is $y' = 2y$
- A solution is $y_h = e^{2t}$
- If we set $y = vy_h$, then $v = \int \frac{q(t)}{y_h} dt = \int \frac{t^2 e^{2t}}{e^{2t}} dt = \int t^2 dt = \frac{1}{3} t^3 + C$
- Since $y = vy_h$, this means $y = \frac{1}{3} t^3 e^{2t} + Ce^{2t}$

Summary!

What we did today:

- We learned about linear equations
- We learned how to solve them with integrating factors
- We learned how to solve them with variation of parameters

Plan for next time:

- More practice solving first order linear equations

Squirrel says:

Whew, all done