Math 307 Lecture 5

Modeling Equations like a Pro!

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Today!

Last time:

- Review of first order linear and separable ODEs

This time:

- First homework due Friday!!!
- Modeling first-order equations

Next time:

- Differences between linear and nonlinear equations

Outline

Our First Example Model: Mixing fluids

Figure: Rate of pollution of a pond can be modeled by a linear ordinary differential equation

- Polluted water flows into a pond
- Volume of pond (constant): $V = 10^7$ gal
- Amount of pollutant in pond: P (metric tons)
- Toxic sludge flows in at 5×10^6 gal/yr
- Toxic sludge contains $2 + \sin(2t)$ grams of pollutant per gallon
- Lake unpolluted at $t = 0$

Our First Example Model: Mixing fluids

Question

How does the amount of pollutant change over time?

$$\frac{dP}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = \overbrace{(5 \times 10^6)}^{\text{gal. toxic sludge/yr}} \cdot \underbrace{(2 + \sin(2t))}_{\text{grams pollutant/gal}} \cdot \overbrace{(10^{-6})}^{\text{grams/ton}}$$

$$\text{rate out} = \overbrace{(5 \times 10^6)}^{\text{gal. mixed pond water/yr}} \cdot \underbrace{P(t)/V}_{\text{metric tons pollutant/gal}}$$

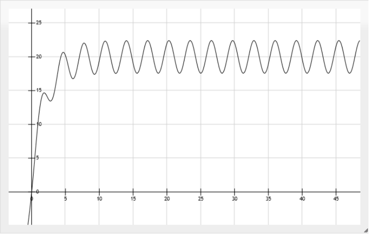
Our First Example Model: Mixing fluids

- IVP $P(0) = 0$ and

$$\frac{dP}{dt} = 10 + 5\sin(2t) - \frac{1}{2}P$$

- Int. factor is $\mu(t) = e^{t/2}$
- Solution given below
- Limit behavior: oscillation about P=20

Figure: Pond pollution modeled by a linear ODE



$$P(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) - \frac{300}{17}e^{-t/2}$$

Our Next Example Model: Radiative Heat Transfer

Figure: Radiative heat

- U abs. temp. of body
- T abs. temp. of space
- α emissivity constant

- Stefan-Boltzmann law:

$$\frac{dU}{dt} = -\alpha(U^4 - T^4)$$

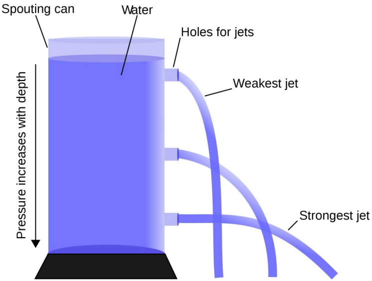
- For $U >> T$, we can approximate

$$\frac{dU}{dt} = -\alpha U^4$$

- Separable! Solution is $U = \frac{1}{(3\alpha t + C)^{1/3}}$

Torricelli's Law

Figure: Water under more pressure shoots faster/farther



- Torricelli's law says
- $v = \sqrt{2gh}$
- Outflow velocity: v
- Water level above opening: h
- How fast does water leave the tank?

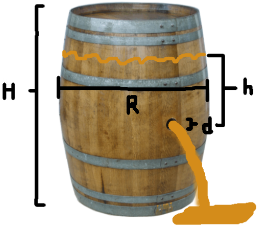
An Old West Example

Figure: “You’ve got to ask yourself one question: ‘Do I feel lucky?’ Well, do ya punk?” – Clint Eastwood on Differential Equations

- Clint Eastwood shoots a cylindrical barrel of whiskey dead center
- bullet does not leave barrel
- barrel has height 3 feet and radius 1 foot
- he uses a .44 Magnum, the most powerful handgun in the world
- how long until the barrel is half-empty?

An Old West Example

Figure: "Glub glub glub..."



- $H = 3$ ft, $R = 1$ ft
- $d = 10.9$ millimeters (from size of bullet)
- V volume of barrell
- area of hole: $A = \pi d/2 = 0.297 \cdot 10^{-4}\pi$ ft
- $dV/dt = -A \cdot v_{\text{out}}$

An Old West Example

- Toricelli says $v_{\text{out}} = \sqrt{2gh}$.
- Therefore
$$\frac{dV}{dt} = -A\sqrt{2gh}$$
- Also since $V = \pi R^2 h$, we have
$$\frac{dV}{dt} = \pi R^2 \frac{dh}{dt}$$
- Therefore
$$-A\sqrt{2gh} = \pi R^2 \frac{dh}{dt}$$

An Old West Example

- Separable equation!

$$\frac{1}{\sqrt{h}}\frac{dh}{dt} = -\frac{A\sqrt{2g}}{\pi R^2}$$

$$2\sqrt{h} = -\frac{A\sqrt{2g}}{\pi R^2}t + C$$

$$h = \left(-\frac{A\sqrt{g}}{\sqrt{2}\pi R^2}t + C\right)^2$$

- At $t = 0$, $h = H/2$, so $C = \sqrt{H/2}$ and

$$h = \frac{1}{2}\left(\sqrt{H} - \frac{A\sqrt{g}}{\pi R^2}t\right)^2$$

An Old West Example

- Set $h = 0$ and solve for t:

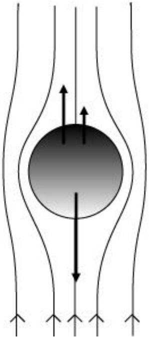
$$t = \frac{\sqrt{H}\pi R^2}{A\sqrt{g}}$$

- Putting in the values of A, H, g ($g = 32$ ft/s^2)

$$t = 8.42 \cdot 10^3 \text{ s} \approx 14 \text{ min.}$$

Stokes's Law

Figure: Falling pebble feels three forces



- Stokes law governs the drag felt by an object falling through a viscous fluid
- Spherical pebble of radius r, mass m, and velocity v
- Ball feels three forces: buoyant force, gravitational force, viscous drag

Stokes's Law

Figure: Stokes was known for being super stoked about fluid mechanics

- buoyant force: B equals weight of fluid displaced
- viscous drag: $R = -6\pi\mu rv$ by Stokes law
- here μ quantifies how viscous the fluid is
- μ is bigger for molasses than for water
- gravitational force: $-mg$
- How fast does a pebble sink?

Figure: The Beatles did math in a yellow submarine...maybe

- We drop a spherical submarine into the ocean
- assume water density is approximately constant with respect to depth
- buoyant force is then $B = \frac{4}{3}\pi r^3 \rho g$
- by Newtons law: $F = m\frac{dv}{dt}$
- total force

$$F = -mg + B + R$$

- velocity therefore satisfies the differential equation

$$m\frac{dv}{dt} = \overbrace{-mg}^{\text{grav. force}} + \underbrace{\frac{4}{3}\pi r^3 \rho g}_{\text{buoyant force}} - \overbrace{6\pi\mu r v}^{\text{viscous drag}}$$

- Linear equation! (alt. it's separable)
- Find an integrating factor and solve. Should get:

$$v = C\exp\left(\frac{-6\pi\mu r}{m}t\right) + \frac{-mg + \frac{4}{3}\pi r^3\rho g}{6\pi\mu r}$$

3000 Leagues under the Sea

Figure: Sea monsters love differential equations

- as we fall, we gather more speed!
- what is the terminal velocity of the submarine (maximum speed it can fall)?
- assume not attacked by a sea monster
- take limit as $t \to \infty$

$$v_{\text{term}} = \frac{-mg + \frac{4}{3}\pi r^3 \rho g}{6\pi\mu r}$$

Summary!

What we did today:

- We looked at some real-world situations that can be modeled by differential equations

Plan for next time:

- Differences between linear and nonlinear equations