Math 307 Lecture 7

Autonomous Equations and Population Dynamics

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Today!

Last time:

- Differences between linear and nonlinear equations

This time:

- Autonomous Equations and Population Dynamics

Next time:

- Numerical Approximations to Solutions

Outline

Definition

Definition

An autonomous equation is a differential equation of the form $y' = f(y)$

Example

Suppose a student takes a loan with interest rate r compounded continuously, and pays back continuously at a rate $k(t)$. Then the amount of money owed S satisfies the differential equation

$$y' = ry - k$$

This equation is autonomous if and only if $k(t)$ is a constant.

Definition

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An first-order *autonomous equation* is a differential equation of the form $y' = f(y)$

Example

Suppose a student takes a loan with interest rate r compounded continuously, and pays back continuously at a rate $k(t)$. Then the amount of money owed S satisfies the differential equation

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This equation is autonomous if and only if $k(t)$ is a constant.

Properties

- Many laws of physics obey autonomous equations, since the basic laws of nature shouldn't change with time
- Easy to solve (separable), though the integral is not always easy to do explicitly!
- Often more interested in asking qualitative questions:
  - what do solutions look like?
  - how quickly do they grow?
  - what is its limit behavior?
  - dependence on initial conditions?
- Have "equilibrium solutions" that solutions tend toward or away from

Definitions

Definition

A constant solution to an autonomous differential equation is called an *equilibrium solution*.

- Consider the general autonomous equation $y' = f(y)$
- What are the stable solutions?
- The stable solutions are exactly the roots of f!
- These are also called the *critical points* of f

Stability of Equilibrium Solutions

- Suppose K is a root of $f(y)$
- Does a solution to the IVP

$$y' = f(y), \quad y(t_0) = y_0$$

  tend toward or away from K as t increases?
- K is stable if solution tends toward
- K is unstable if solution tends away
- K is semistable, if it is a combination of both

Example: Logistic Equation

- One model for population growth is the so-called *logistic equation*

$$y' = r\left(1 - \frac{y}{K}\right) y$$

- r is called the *intrinsic growth rate* (must be postive!)
- K is called the *environmental carrying capacity* (postive!)
- The equilibrium solutions are $y = 0$ and $y = K$

Example: Logistic Equation

- Want to find a solution to the equation satisfying $y(0) = y_0$
- If $y_0 = 0$, then $y = 0$
- If $y_0 > 0$, then y approaches K for large t
- If $y_0 < 0$, then y goes to $-\infty$
- All this, we can see from the slope field, even before solving the equation!
- However, solution is

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Example: Critical Threshold

- Last model does not allow populations to die out
- Another model for population growth related to the logistic equation is

$$y' = -r\left(1 - \frac{y}{T}\right)y$$

- Here again r, T are positive constants
- Here again T is called the *critical amplitude*
- Not much difference in the equation (just a minus sign)
- Equilibrium solutions are $y = 0$ and $y = T$
- If $y_0 > T$, population increases exponentially
- If $0 < y_0 < T$, population dies out

Example: Logistic with a Critical Threshold

- Last equation forces population to either die out or grow without bound
- For population growth, this seems unreasonable
- Fixed by introducing a “hybrid model"

$$y' = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

- Here again r, T, K are positive constants and $T < K$
- Equilibrium solutions are $y = 0, y = T$ and $y = K$
- If $T < y_0$, population increases toward K
- If $0 < y_0 < T$, population dies out

Review!

Today:

- Autonomous Equations and Population Dynamics

Next time:

- Numerical solutions to differential equations