Math 307 Lecture 8

Numerical Approximations of Solutions to IVPs

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Today!

Last time:

- Integrating Factors

This time:

- Numerical Approximations

Next time:

- Exam!

Outline

1. Numerical Approximations
   - What is a Numerical Approximation
   - Numerical Approximations to IVPs

2. Euler's Method
   - What is Euler's Method
   - Examples

A First Look at Approximations

- Q: How can we approximate a solution to an IVP?

$$y' = f(t, y), \quad y(t_0) = y_0$$

- Q: More generally, how can we approx. any function...?
- A: as a table of values!
- For example, we can approximate $y(t) = t^2$ on the interval $[0, 1]$ as a table of values, eg.

t	y(t)
0.0	0.00
0.2	0.04
0.4	0.16
0.6	0.36
0.8	0.64
1.0	1.00

A First Look at Approximations

- The table can be thought of as representing a piecewise linear function
- What does that mean? Well think about this question:
- Q: Using the previous approximation of $y(t) = t^2$, what is the value of $y(0.1)$ *according to our approximation*?
- A: To figure this out, we use *linear interpolation*:
  - For t between 0.0 and 0.1, our approximation says $y(t)$ behaves like the line between $(0.0, 0.00)$ and $(0.2, 0.04)$.
  - $y(t) \approx 0.2(t - 0.0) + 0.00$ for $0 \leq t \leq 0.02$
  - so in particular $y(0.1) \approx 0.02$
- thus a table represents a function as a piecewise linear function

Approximations to $y(t) = t^2$

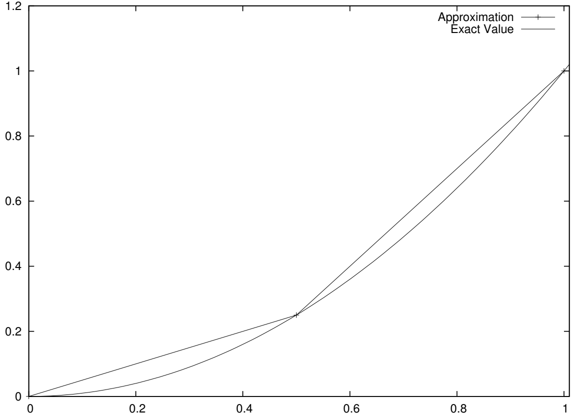
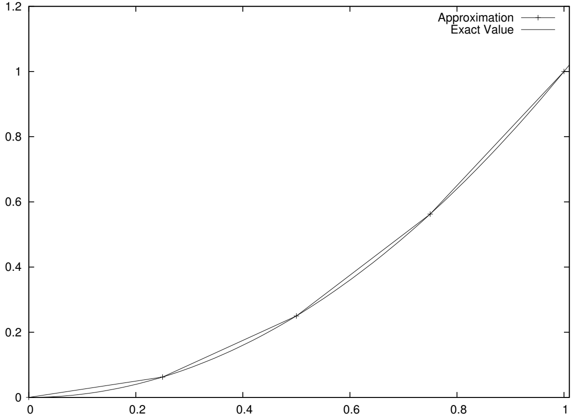


Figure: Approximation of $y(t) = t^2$ with spacing of 0.50

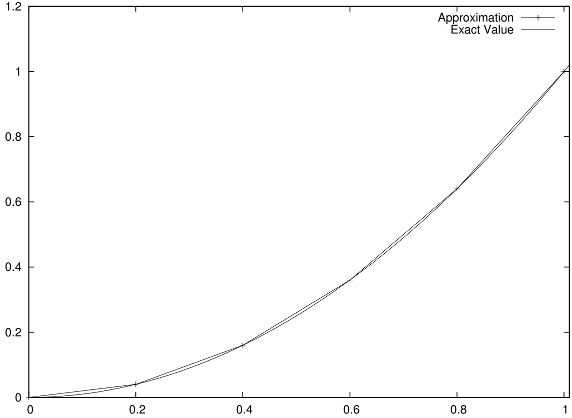
Approximations to $y(t) = t^2$

Figure: Approximation of $y(t) = t^2$ with spacing of 0.25



Approximations to $y(t) = t^2$

Figure: Approximation of $y(t) = t^2$ with spacing of 0.20



Some Observations

- As the number of entries in the table increases, the approximation improves!
- The approximation is completely accurate at points in the table, but we shouldn't require this in general
- We could do something similar for a much more complicated function than $y(t) = t^2$
- You may have wondered how computers create plots of functions... this is exactly how!

That Well and Good but What about IVPs?

- We just approximated $y(t) = t^2$, but how to we approximate a solution to an IVP without actually solving it?
- Let's start out with an example!

Example

Approximate a solution to the IVP

$$y' = y, \quad y(0) = 1.$$

- We know how to solve this exactly, but let's pretend that we don't.
- How can we generate a table of entries that approximate a solution?

An Iterative Method

- Let's pick a spacing for our table beforehand: $\Delta t = 0.1$
- There's an immediately obvious entry we should have in our table

t	y(t)
0.0	1.00

- What should be the next entry?
- We know that $y'(0) = 1$ (why??), so by linear approx.

$$y(t) \approx y'(0)(t - 0.0) + 1.00 = t + 1$$

- In particular $y(0.1) \approx 1.10$

t	y(t)
0.0	1.0000
0.1	1.1000

An Iterative Method

- What about the next data point in the table?
- If $y(0.1) \approx 1.10$, then $y'(0.1) \approx 1.10$ (why??)
- Thus by linear approx for t close to 0.1:

$$y(t) \approx y'(0.1)(t - 0.1) + 1.10 = 1.1t + 0.99$$

- In particular $y(0.2) \approx 1.21$

t	y(t)
0.0	1.0000
0.1	1.1000
0.2	1.2100

An Iterative Method

- What about the next data point in the table?
- If $y(0.2) \approx 1.21$, then $y'(0.2) \approx 1.21$ (why??)
- Thus by linear approx for t close to 0.2:

$$y(t) \approx y'(0.2)(t - 0.2) + 1.21 = 1.21t + 0.968$$

- In particular $y(0.3) \approx 1.331$

t	y(t)
0.0	1.0000
0.1	1.1000
0.2	1.2100
0.3	1.3310

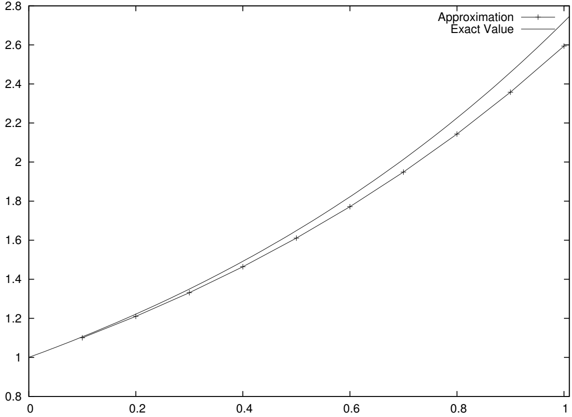
An Iterative Method

- Continuing in this fashion, we get

t	y(t)
0.0	1.0000000000
0.1	1.1000000000
0.2	1.2100000000
0.3	1.3310000000
0.4	1.4641000000
0.5	1.6105100000
0.6	1.7715610000
0.7	1.9487171000
0.8	2.1435888100
0.9	2.3579476910
1.0	2.5937424601

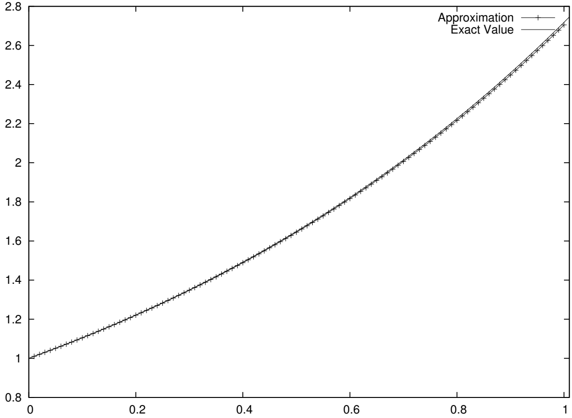
An Iterative Method

Figure: Approx. Soln of $y'(t) = y(t)$ with $y(0) = 1$ and $\Delta t = 0.1$



An Iterative Method

Figure: Approx. Soln of $y'(t) = y(t)$ with $y(0) = 1$ and $\Delta t = 0.01$



A Couple Comments

- Q: As t moves farther away from the initial time t_0, what happens to the accuracy of the solution?
- A: It becomes less accurate! (Can you think of why this might be?)
- Q: If we choose smaller time steps Δt, will the accuracy go up or down?
- A: As we decrease Δt, we should expect the solution to be more accurate for longer (why?)
- Q: Why do we care to approximate solutions?
- A: Most first-order equations in the "wild" are not solvable explicitly!!

The Method

- We want to “abstractify" our method of approximation, so that it can be applied to any situation
- Given an initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

- For some choice of n and Δt, we want to find an approximate solution on an interval $[t_0, t_0 + n\Delta t]$

The Method

- This is done by Euler's Method:
- *Euler's Method* gives us a piecewise linear approximation defined by the table of values

t	$y(t)$
t_0	y_0
t_1	y_1
$\vdots$	$\vdots$
t_n	y_n

- where for $j > 0$, $t_j = t_0 + j \cdot \Delta t$, and

$$y_j = f(t_{j-1}, y_{j-1}) \cdot \Delta t + y_{j-1}$$

Solving the IVP $y' = \sin(ty), \;\; y(0) = 1$

- To make this idea clearer, let's look at some more examples of approximating solutions with Euler's Method:

Example

Find an approximate solution to the IVP

$$y' = \sin(ty), \;\; y(0) = 1$$

on the interval $[0, 10]$ using a spacing of $\Delta t = 0.01$

- This equation is too hard to solve explicitly, since it's not linear, separable, homogeneous, etc.
- We can find an approximate solution with Euler's Method!

Solving the IVP $y' = \sin(ty), \quad y(0) = 1$

Computer Code (Python)

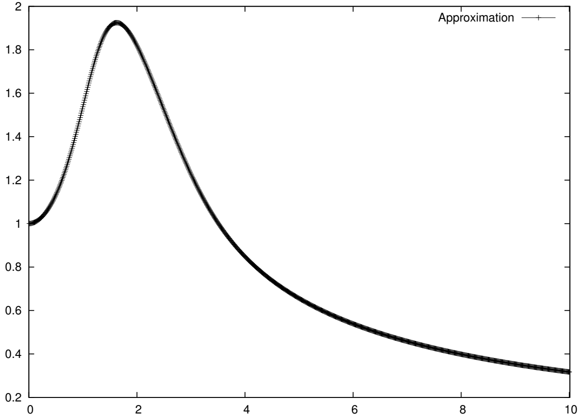
```
#!\usr\bin\python
import math

dx = 0.01
nsteps = 1000
x = 0 # starting point
y = 1 # initial condition

for i in range(0,nsteps):
  x = x + dx
  dy = math.sin(x*y)
  y = dy*dx + y
  print x, y
```

Graph of Approximate Solution to the IVP

Figure: Approx. Soln of $y'(t) = \sin(ty)$ with $y(0) = 1$ and $\Delta t = 0.01$



Review!

Today:

- Numerical solutions to differential equations

Next time:

- Exam!