Weekly Homework 1

Due: Monday April 13, 2015

January 16, 2016

Problem 1 (Solutions To Differential Equations). For each of the following , show whether or not the specified function is a solution to the corresponding differential equation.

(a) $y'''' + y''' + y' - y = 0$, $y(x) = \cos(x)$

(b) $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0$, $u(x,t) = \frac{1}{2}c\text{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - a)\right]$

(c) $y'' - y = 0$, $y(x) = \sinh(x)$

Solution 1.

(a) $y' = -\sin(x)$, $y''' = \sin(x) = -y'$, $y'''' = \cos(x) = y$. Therefore $y''' + y''' + y' - y = 0$

(b) This is a famous equation known as the KdV equation. The function $u(x,t)$ is a well-known solution, called a soliton solution. To see that it is a solution, one can take all the various partial derivatives of $u(x,t)$ and plug everything in. Alternatively, one may define $z = \frac{\sqrt{c}}{2}(x - ct - a)$. Then $u(x,t) = (c/2)f(z)$ for $f(z) = \text{sech}^2(z)$. Therefore

$$u_t = -(c^{5/2}/4)f'(z), \quad u_x = (c^{3/2}/4)f'(z), \quad u_{xxx} = (c^{5/2}/16)f'''(z).$$

Substituting this in to the KdV equation, we obtain

$$-(c^{5/2}/4)f'(z) + (c^{5/2}/16)f'''(z) + 6(c^{5/2}/8)f'(z) = 0.$$

Multiplying both sides by $16/c^{5/2}$, this becomes

$$-4f'(z) + f'''(z) + 12f(z)f'(z) = 0.$$

Thus we need only show that $f(z) = \text{sech}^2(z)$ satisfies the above equation. We calculate

$$f'(z) = 2\text{sech}^2(z)\tanh(z)$$

$$\begin{aligned} f''(z) &= 2(\text{sech}^2(z))'\tanh(z) + 2\text{sech}^2(z)(\tanh(z))' \\ &= 2(2\text{sech}^2(z)\tanh(z))\tanh(z) + 2\text{sech}^2(z)(-\text{sech}^2(z)) \\ &= 4\text{sech}^2(z)\tanh^2(z) - 2\text{sech}^4(z) \\ &= 4\text{sech}^2(z)(1 - \text{sech}^2(z)) - 2\text{sech}^4(z) \\ &= 4\text{sech}^2(z) - 6\text{sech}^4(z). \end{aligned}$$

$$
\begin{aligned}
f'''(z) &= 8\text{sech}(z)(\text{sech}(z))' - 24\text{sech}^3(z)(\text{sech}(z))' \\
&= 8\text{sech}^2(z)\tanh(z) - 24\text{sech}^4(z)\tanh(z).
\end{aligned}
$$

Thus

$$
\begin{aligned}
-4f'(z) + f'''(z) + 12f(z)f'(z) =& -4(2\text{sech}^2(z)\tanh(z)) \\
&+ (8\text{sech}^2(z)\tanh(z) - 24\text{sech}^4(z)\tanh(z)) \\
&+ 12(\text{sech}^2(z))(2\text{sech}^2(z)\tanh(z)) \\
=& 0.
\end{aligned}
$$

Thus f satisfies the equation, and it follows that $u(x,t)$ is a solution to the KdV equation

(c) Note that $y' = \cosh(x)$ and $y'' = \sinh(x) = y$. Thus $y'' - y = 0$

Problem 2 (Solving differential equations). For each of the following differential equations, do the following

(i) Identify the type of differential equation

(ii) Find the "general solution"

(a) $y' = 2y + 3$

(b) $y' = \frac{x^2-y^2}{x+y}$

(c) $\sin(u)\frac{du}{dt} = \cos(u)/(1+t^2)$

(d) $\frac{dy}{dt} = \frac{t^2-y^2}{ty}$

(e) $(3x - 4y)dy = (2x + 7y)dx$

(f) $\frac{dy}{dt} + y/t = 6\cos(4t)$

(g) $y' + y = \cos(t)$

(h) $y' = 1 - y^3$

Solution 2.

(a) This equation is linear, with integrating factor $\mu(x) = e^{-2x}$. Thus

$$e^{-2x}y' - 2e^{-2x}y = 3e^{-2x}$$

is exact. Grouping things together, we obtain

$$(e^{-2x}y)' = 3e^{-2x}$$

and therefore

$$e^{-2x}y = -\frac{3}{2}e^{-2x} + C.$$

Thus

$$y = -\frac{3}{2} + Ce^{2x}.$$

(b) This equation is linear since it simplifies to

$$y' = x - y$$

An integrating factor for this equation is e^x, giving us the exact equation

$$e^x y' + e^x y = xe^x.$$

Grouping things together, we obtain

$$(e^x y)' = xe^x.$$

Integrating, we now obtain

$$e^x y = xe^x - e^x + C.$$

Therefore

$$y = x - 1 + Ce^{-x}.$$

(c) This equation is separable. Separating, we obtain

$$\tan(u)du = \frac{1}{1+t^2}dt.$$

Now integrating, we obtain

$$-\ln\cos(u) = \tan^{-1}(t) + C.$$

Therefore

$$u = \cos^{-1}(\exp(-\tan^{-1}(t) + C)).$$

(d) this equation is homogeneous, since it simplifies to

$$\frac{dy}{dt} = (y/t)^{-1} - (y/t).$$

Using the substition $z = y/t$, $y' = z + tz'$ we then obtain

$$z + tz' = z^{-1} - z.$$

Simpliflying this equation, it becomes

$$tz' = \frac{1 - 2z^2}{z}.$$

This is separable! Separating, we obtain

$$\frac{z}{1-2z^2}dz = \frac{1}{t}dt.$$

Now integrating, we obtain

$$-\frac{1}{4}\ln(1 - 2z^2) = \ln(t) + C.$$

Solving for z, we obtain

$$z = \pm\sqrt{Ct^{-4} + 1/2}.$$

Then since $y = tz$, it follows that

$$y = \pm t\sqrt{Ct^{-4} + 1/2}.$$

(e) This equation is homogeneous, since we may simplifiy it to

$$y' = \frac{2x+7y}{3x-4y} = \frac{2+7(y/x)}{3-4(y/x)}.$$

Then doing the substitution $z = y/x$, $y' = z + xz'$, we obtain

$$z + xz' = \frac{2+7z}{3-4z}.$$

This simplifies to

$$xz' = \frac{2+4z-4z^2}{3-4z}.$$

This is separable! Separating, we obtain

$$\frac{3-4z}{2+4z-4z^2}dz = \frac{1}{x}dx.$$

Integrating the left hand side, we get

$$\begin{aligned}
\int \frac{3-4z}{2+4z-4z^2}dz &= \int \frac{1}{2+4z-4z^2}dz + \int \frac{2-4z}{2+4z-4z^2}dz \\
&= \int \frac{1/4}{3/4-(z-1/2)^2}dz + \int \frac{2-4z}{2+4z-4z^2}dz \\
&= \frac{1}{2\sqrt{3}}\tanh^{-1}\left(\frac{2}{\sqrt{3}}(z-1/2)\right) + \frac{1}{2}\ln|2+4z-4z^2| + C
\end{aligned}$$

and thus

$$\frac{1}{2\sqrt{3}}\tanh^{-1}\left(\frac{2}{\sqrt{3}}(z-1/2)\right) + \frac{1}{2}\ln|2+4z-4z^2| = \ln|x| + C.$$

(f) This equation is linear. An integrating factor is $\mu(t) = t$. Therefore the equation

$$ty' + y = 6t\cos(4t)$$

is exact. Grouping terms, we find

$$(ty)' = 6t\cos(4t)$$

Integrating both sides, it follows that

$$ty = \frac{3}{2}t\sin(4t) - \frac{3}{8}\cos(4t) + C$$

Therefore

$$y = \frac{3}{2}\sin(4t) - \frac{3}{8}t^{-1}\cos(4t) + Ct^{-1}.$$

(g) This equation is linear. An integrating factor is e^t. Therefore the equation

$$e^t y' + e^t y = e^t \cos(t)$$

is exact. Grouping terms we obtain

$$(e^t y)' = e^t \cos(t).$$

Integrating both sides, it follows that

$$e^t y = \frac{1}{2} e^t \sin(t) + \frac{1}{2} e^t \cos(t) + C$$

Therefore

$$y = \frac{1}{2}\sin(t) + \frac{1}{2}\cos(t) + Ce^{-t}.$$

(h) The equation is separable. Separating it, we obtain

$$\frac{1}{1-y^3} y' = 1.$$

To integrate this equation, we must use partial fraction decomposition. We find

$$\frac{1}{1-y^3} = \frac{A}{1-y} + \frac{By+C}{1+y+y^2}$$

Clearing denominators, we obtain

$$1 = A(1+y+y^2) + (By+C)(1-y).$$

When $y = 1$, this shows $1 = 3A$, so $A = 1/3$. When $y = 0$, this shows $1 = A + C$, and therefore $C = 2/3$. Comparing coefficients of y^2, we also see that $A = B$, and therefore $B = 1/3$. Thus

$$\frac{1}{1-y^3} = \frac{1/3}{1-y} + \frac{1/3y + 2/3}{1+y+y^2}$$

Therefore

$$\begin{aligned}
\int \frac{1}{1-y^3} dy &= \int \frac{1/3}{1-y} dy + \int \frac{1/3y+2/3}{1+y+y^2} dy \\
&= \int \frac{1/3}{1-y} dy + \int \frac{1/3y+1/6}{1+y+y^2} dy + \int \frac{1/2}{1+y+y^2} dy \\
&= \int \frac{1/3}{1-y} dy + \int \frac{1/3y+1/6}{1+y+y^2} dy + \int \frac{1/2}{3/4+(y+1/2)^2} dy \\
&= -\frac{1}{3}\ln|1-y| + \frac{1}{3}\ln|1+y+y^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(y+1/2)\right) + C
\end{aligned}$$

Therefore

$$-\frac{1}{3}\ln|1-y| + \frac{1}{3}\ln|1+y+y^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(y+1/2)\right) = x + C.$$

Problem 3 (Waaaaait a minute!). Explain what is wrong with the following argument:

> Consider the differential equation
> $$y' = 1 - 2y$$
> Integrating both sides, we get the equation
> $$y = y - y^2 + C.$$
> Simplifying this, we get the solution $y^2 = C$ meaning that
> $$y = \pm\sqrt{C}.$$

Solution 3. The problem with this "solution" is that the person integrated the function of y with respect to x. In particular

$$\int 1 - 2y dx \neq \int 1 - 2y dy = y - y^2 + C,$$

just as

$$\int y' dy \neq \int y' dx = y + C.$$

Thus the whole argument is garbage from the beginning.

Problem 4 (Slope fields). For each of the following initial value problems

(i) Plot the slope field

(ii) Based on the plot of the slope field, predict the behavior of a solution to the IVP at large values of t

(iii) Explicilty solve the IVP

(iv) Based on the explicit solution of the IVP, determine the behavior at large values of t

(a) $y' = y(1 - y^2)$, $y(0) = 1$

(b) $y' = y(1 - y^2)$, $y(0) = 1/2$

(c) $y' = y(1 - y^2)$, $y(0) = 3/2$

Solution 4. The equation $y' = y(1 - y^2)$ is separable. Solving it in the usual fashion, we obtain the family of solutions

$$\frac{y}{\sqrt{1 - y^2}} = Ce^x.$$

How can we solve for y here? Squaring, we obtain

$$\frac{y^2}{1 - y^2} = Ce^{2x}.$$

Multiplying by $1 - y^2$ on both sides, this becomes

$$y^2 = Ce^{2x} - y^2 Ce^{2x}.$$

Therefore

$$y^2(1 + Ce^{2x}) = Ce^{2x},$$

making

$$y^2 = \frac{Ce^{2x}}{1 + Ce^{2x}}.$$

Thus

$$y = \pm\sqrt{\frac{Ce^{2x}}{1 + Ce^{2x}}}.$$

(a) Note that the family of solutions that we found does not contain a particular solution to this IVP. However, a solution does exist! In particular $y = 1$ is a solution. Based on the slope field, this makes a great deal of sense!

(b) An explicit solution is given by

$$y = \sqrt{\frac{e^{2x}}{1 + e^{2x}}}.$$

As $x \to \infty$, this shows that $y \to 1$, which agrees well with the picture of the slope field.

(c) An explicit solution is given by

$$y = \sqrt{\frac{3e^{2x}}{3e^{2x} - 1}}.$$

As $x \to \infty$, this shows that $y \to 1$, which agrees well with the picture of the slope field.

Problem 5 (Second order equations). Consider the second order differential equation

$$y'' - y = 0$$

(a) Show that the change of variables $z = y' + y$ in the above second-order equation transforms it into the first order equation

$$z' - z = 0$$

(b) Find the general solution of the first-order equation of (a)

(c) By substituting the value of z back into the equation $z = y' + y$, find the value of y. Your final answer for y should involve *two* arbitrary constants.

Solution 5.

(a) If $z = y' + y$, then $z' = y'' + y'$, and therefore $y'' - y = (z' - y') - y = z' - z$. Thus the second order equation becomes the first order equation $z' - z = 0$.

(b) The equation of (a) is separable. The general solution is $z = Ae^x$, where A is an arbitrary constant.

(c) Since $z = y' + y$, this means $y' + y = Ae^x$. This is a first order linear equation with integrating factor e^x. Therefore the equation $e^x y' + e^x y = Ae^{2x}$ is exact. Grouping, we obtain $(e^x y)' = Ae^{2x}$. Therefore $e^x y = Ae^{2x} + B$. It follows that

$$y = Ae^x + Be^{-x},$$

where A and B are both arbitrary constants. Note that the general solution of the second order equation that we just found involves *two* arbitrary constants, instead of just one.

Problem 6 (Solving Initial Value Problems). Find a solution to each of the following initial value problems

(a) $y' = x\cos(y)$, $y(0) = 1$

(b) $y' = e^x + y$, $y(1) = 2$

(c) $\frac{dy}{dt} + 2y = te^{-2t}$, $y(1) = 0$

(d) $xy' + 2y = \sin(x)$, $y(\pi/2) = 1$

Solution 6.

(a) This is separable. Separating, we obtain

$$\sec(y)dy = xdx.$$

Integrating, we obtain

$$\ln|\sec(y) + \tan(y)| = \frac{1}{2}x^2 + C.$$

Then substituting in 1 for y and 0 for x, we get $C = \ln|\sec(1) + \tan(1)|$, and therefore our particular solution is

$$\ln|\sec(y) + \tan(y)| = \frac{1}{2}x^2 + \ln|\sec(1) + \tan(1)|.$$

Note that in this case it is too difficult to solve for y in terms of x.

(b) This equation is linear with integrating factor $\mu(x) = e^{-x}$. Therefore we have an exact equation

$$e^{-x}y' - e^{-x}y = 1$$

Grouping and integrating, we obtain

$$e^{-x}y = x + C.$$

Therefore the general solution is

$$y = xe^{-x} + Ce^{-x}.$$

Using the initial condition $y(1) = 2$, we get $C = 2e - 1$. Thus

$$y = xe^{-x} + (2e - 1)e^{-x}.$$

(c) This equation is linear, with integrating factor e^{2t}. Solving it in the usual way, we obtain the general solution

$$y = \frac{1}{2}t^2 e^{-2t} + Ce^{-2t}.$$

The initial condition $y(1) = 0$ then tells us $C = -\frac{1}{2}$. Thus

$$y = \frac{1}{2}(t^2 - 1)e^{-2t}.$$

(d) This equation is linear, with integrating factor x. Solving it in the usual way, we obtain

$$y = -x^{-1}\cos(x) + x^{-2}\sin(x) + Cx^{-2}.$$

The initial condition $y(\pi/2) = 1$ then tells us $C = \pi^2/4 - 1$. Therefore the particular solution is

$$y = -x^{-1}\cos(x) + x^{-2}\sin(x) + \left(\frac{\pi^2}{4} - 1\right)x^{-2}.$$

Problem 7 (An almost homogeneous equation). Consider the differential equation

$$y' = x\cos(y/x) + y/x$$

(a) Explain why this is not a homogeneous differential equation

(b) Find the general solution of the differential equation.

Solution 7.

(a) The right hand side is not a function of y/x only because of the extra factor of x multiplying $\cos(y/x)$.

(b) Even though it isn't homogeneous, we can still try the substitution $z = y/x$ and $y' = z + xz'$. Doing so, we obtain the equation

$$z + xz' = x\cos(z) + z.$$

This simplifies to

$$z' = \cos(z).$$

This is separable, with solution

$$\ln|\sec(z) + \tan(z)| = x + C.$$

Then using $z = y/x$, we obtain

$$\ln|\sec(y/x) + \tan(y/x)| = x + C.$$

Note that in this case it is too difficult to find y in terms of x only.