MATH 307: Problem Set #2

Due on: Jan 27, 2016

Problem 1 *Exact Equations*

In each of the following, determine if the equation is exact. If it is exact, then find the solution.

(i) $(2x + 4y) + (2x - 2y)y' = 0$

(ii) $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$

(iii) $\frac{dy}{dx} = -\frac{ax-by}{bx-cy}$

(iv) $(e^x \sin y + 3y)dx - (3x - e^x \sin y)dy = 0$

(v) $(y/x + 6x)dx + (\ln(x) - 2)dy = 0$

(vi) $\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}} = 0$

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Problem 2 **Fluid Mixing**

A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gallons while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t.

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Problem 3 More Fluid Mixing

Initially, a mass of ten grams of salt is dissolved in a 10 liter tank full of water. Then water containing salt at a concentration of 10 grams per liter trickles in at a rate of two liters per hour. A well-mixed solution trickles out at a rate of 3 liters per hour. Find the concentration (in grams per liter) of the salt in the tank at the time when the tank contains 4 liters.

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Problem 4 Monetary Investment

A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

(a) Determine the sum $S(t)$ accumulated at any time t

(b) If $r = 7.5\%$ determine k so that 1 million will be available for retirement in 40 years

(c) If $k = 2000$ per year, determine the return rate r that must be obtained to have 1 million available in 40 years

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Problem 5 More Fluid Mixing

A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos(t))$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

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Problem 6 Whale Fall

The expression "whale fall" refers to the body of a deceased whale which has fallen to the ocean floor. Suppose that a whale dies of old age (after living a long and happy life, so there's nothing to blubber about). The whale immediately begins to sink, from rest at its initial position on the surface. The whale has mass m and cross-sectional area a.

(i) the density of ocean water is $\rho = 1027$ kg/m^3

(ii) the gravitational acceleration is $g = 9.81$ m/s^2

(iii) the force of drag satisfies the "drag equation" $F_D = \frac{1}{2}\rho u^2 c_d a$ where here a is the cross-sectional area and c_d is the coefficient of drag

(iv) the whale's body descends to the ocean floor, h meters down, unmolested by other life

With this in mind, answer the following questions

(a) set up a first-order initial value problem describing the speed u of the carcass as a function of time

(b) show that $u(t) = K \tanh(gt/K)$ is a solution to this initial value problem, where here $K = \sqrt{2mg/(\rho c_d A)}$ is called the *terminal velocity*, which is the maximum speed of the falling body

(c) find an equation, in terms of K and h, for how long it takes the whale to reach the ocean floor

(d) for a blue whale, we may approximate $a = 28$ meters, $m = 122$ tonnes, and $c_d = 0.75$. If the depth of the ocean is $h = 2$ km, how long will the descent take?

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Problem 7 Jean Wilder's Famous Problem

A population of Oompa Loompas in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks time. Also on any given day there is a net migration into the area of 15 Oompa Loompas and 16 are eaten by Wangdoodles, Hornswogglers, Snozzwangers and rotten, Vermicious Knids and 7 die of natural causes. If there are initially 100 Oompa Loompas in the area, will the population survive? If not, when do they die out?

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Problem 8 Bernoulli Equations

A Bernoulli equation is a nonlinear equation of the form

$$y' + p(t)y = q(t)y^n$$

If $n \neq 0$ and $n \neq 1$, then substituting $u = y^{1-n}$ and differentiating yeilds

$$u' = (1-n)y^{-n}y'.$$

This tells us that $y' = \frac{y^n}{(1-n)}u'$. Putting this back into the original differential equation then says

$$\frac{y^n}{1-n}u' + p(t)y = q(t)y^n.$$

Dividing both sides by y, we then get

$$\frac{y^{n-1}}{1-n}u' + p(t) = q(t)y^{n-1}.$$

Now if we notice that $y^{n-1} = 1/u$, then this means

$$\frac{1/u}{1-n}u' + p(t) = q(t)(1/u),$$

which simplifies to

$$\frac{1}{1-n}u' + p(t)u = q(t),$$

which is a linear equation in u. We've just made a nonlinear equation into a linear equation... a small miracle. We can then solve for u, and then use the fact that $u = y^{1-n}$ to obtain y. Let's call this method "Bernoulli's method".

(a) Use Bernoulli's method to solve the differential equation

$$y' = (\Gamma\cos(t) + T)y - y^3$$

where here Γ and T are constants. This equation comes up in the study of stability in fluid flows.

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Problem 9 Norton's Dome

Norton's Dome is a radially symmetric surface whose height above the ground is of the form

$$h(r) = -\frac{2K}{3g}r^{3/2}$$

where r is the radial distance from the center of the dome and our coordinate system is chosen so that the top of the dome has height $h = 0$. here K is a proportionality factor, so that (K/g) has units of length. Set a point mass on top of the dome and let it slide down from the force of gravity, assuming that there are no friction forces. From the laws of classical mechanics, the radial position $r(t)$ of the point mass may be shown to satisfy the initial value problem

$$r'' = K\sqrt{r}, \quad r(0) = 0, \quad r'(0) = 0.$$

(a) Show that $r(t) = K^2t^4/144$ is a solution to the initial value problem

(b) Show that for any $\ell \geq 0$, the function

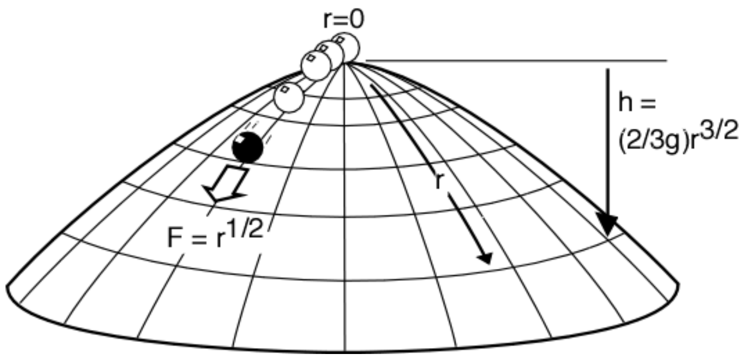

Figure 1: A picture of Norton's Dome.

$$r(t) = \begin{cases} 0, \; t < \ell \\ K^2(t-\ell)^4/144, \; t \geq \ell \end{cases}$$

is also a solution to the initial value problem – hence the initial value problem has more than one solution! This is an example of what is called non-determinism in classical mechanics. There is no unique solution to the initial value problemn: from the point of view of our differential equation, the point particle could simply sit there for all eternity, or it could sit there for some arbitrary amount of time and then suddenly roll off for no particular reason! It's tempting to overstate the meaning of this – for us, this example here is merely to point out that the question of uniqueness of solutions to IVPs may have a negative answer, even in a context that at first blush appears physically relevent.

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