MATH 307: Problem Set #2

Due on: Jan 27, 2016

Problem 1 *Exact Equations*

In each of the following, determine if the equation is exact. If it is exact, then find the solution.

(i) $(2x + 4y) + (2x - 2y)y' = 0$

(ii) $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$

(iii) $\frac{dy}{dx} = -\frac{ax-by}{bx-cy}$

(iv) $(e^x \sin y + 3y)dx - (3x - e^x \sin y)dy = 0$

(v) $(y/x + 6x)dx + (\ln(x) - 2)dy = 0$

(vi) $\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}} = 0$

....

Solution 1.

(i) $M_y = 4$ but $N_x = 2$, so this is not exact

(ii) $M_y = 4xy + 2$ and $N_x = 4xy + 2$, so this is exact. Therefore there exists a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Thus

$$\psi(x, y) = \int (2xy^2 + 2y)\partial x = x^2y^2 + 2xy + h(y)$$

for some unknown function h. Then

$$\psi_y = \frac{\partial}{\partial y}(x^2y^2 + 2xy + h(y)) = 2x^2y + 2x + h'(y),$$

and since $\psi_y = N$, we must have

$$2x^2y + 2x + h'(y) = 2x^2y + 2x.$$

Hence $h'(y) = 0$, meaning that h is a constant which we can take to be zero. Thus $\psi(x, y) = x^2y^2 + 2xy$, and the solution to to the original differential equation is $\psi(x, y) = C$, ie.

$$x^2y^2 + 2xy = C$$

where C is an arbitrary constant.

(iii) We rewrite this equation as

$$\frac{ax - by}{bx - cy} + y' = 0$$

Then $M_y = \frac{(ac-b^2)x}{(bx-cy)^2}$ and $N_x = 0$, so this is not exact

(iv) $M_y = e^x \cos(y) + 3$ and $N_x = -3 + e^x \sin(y)$, so this is not exact

(v) $M_y = 1/x$ and $N_x = 1/x$, so this is exact Therefore there exists a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Thus

$$\psi(x, y) = \int (\ln(x) - 2)\partial y = y\ln(x) - 2y + g(x)$$

for some unknown function g. Then

$$\psi_x = \frac{\partial}{\partial x}(y\ln(x) - 2y + g(x)) = y/x + g'(x),$$

and since $\psi_x = M$, we must have

$$y/x + g'(x) = y/x + 6x$$

Hence $g'(x) = 6x$, so we can take $g(x) = 3x^2$. Thus $\psi(x, y) = y\ln(x) - 2y + 3x^2$, and the solution to to the original differential equation is $\psi(x, y) = C$, ie.

$$y\ln(x) - 2y + 3x^2 = C$$

where C is an arbitrary constant.

(vi) $M_y = \frac{-3xy}{(x^2+y^2)^{5/2}} = N_x$, so this is exact. Therefore there exists a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Thus

$$\psi(x, y) = \int \left(\frac{x}{(x^2 + y^2)^{3/2}}\right) \partial x = \frac{-1}{(x^2 + y^2)^{1/2}} + h(y)$$

for some unknown function h. Then

$$\psi_y = \frac{\partial}{\partial y}\left(\frac{-1}{(x^2 + y^2)^{1/2}} + h(y)\right) = \frac{y}{(x^2 + y^2)^{3/2}} + h'(y),$$

and since $\psi_y = N$, we must have

$$\frac{y}{(x^2 + y^2)^{3/2}} + h'(y) = \frac{y}{(x^2 + y^2)^{3/2}}$$

Hence $h'(y) = 0$, so h is constant, and we can take it to be zero. Thus $\psi(x, y) = \frac{-1}{(x^2+y^2)^{1/2}}$, and the solution to to the original differential equation is $\psi(x, y) = C$, ie.

$$\frac{-1}{(x^2 + y^2)^{1/2}} = C$$

where C is an arbitrary constant.

Problem 2 *Fluid Mixing*

A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gallons per hour while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t.

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Solution 2. We will use P to represent the amount of pollutant (in ounces) in the tank, t the time (in hours), and V the volume of liquid in the tank (in gallons). Initially, the differential equation for the amount P of pollutant in the tank is given by

$$\frac{dP}{dt} = \overbrace{3\frac{\text{gal}}{\text{hr}} \cdot 5\frac{\text{oz}}{\text{gal}}}^{\text{rate in}} - \overbrace{3\frac{\text{gal}}{\text{hr}} \cdot \frac{P}{V}\frac{\text{oz}}{\text{gal}}}^{\text{rate out}}.$$

and satisfies the initial condition that $P(0) = 2$ ounces. Note that the rate in of liquid is equal to the rate out of liquid during the first time period, and therefore $V = V(0) = 800$. Thus the initial value proble we must solve is

$$\frac{dP}{dt} = 15 - \frac{3}{800}P, \; P(0) = 2.$$

The differential equation is separable, and solving it we find

$$P = 4000 + Ce^{-(3/800)t}.$$

The initial condition then tells us that $C = -3998$, and consequently

$$P(t) = 4000 - 3998e^{-(3/800)t}.$$

Next we wish to find the time when the amount of pollutant in the tank is 500 ounces of pollutant. To do so, we set $P(t) = 500$ and solve for t:

$$500 = 4000 - 3998e^{-(3/800)t} \;\Rightarrow\; -(3/800)t = -\frac{800}{3}\ln\frac{3500}{3998} \approx 35.475 \text{ hours}.$$

After this time, the situation in the tank changes. The inflow of pollutant is shut off, and instead fresh water is let in. The inflow rate is no longer the outflow rate, so the volume is not constant. In fact, the initial value problem describing the volume is

$$\frac{dV}{dt} = 2\frac{\text{gal}}{\text{hr}} - 4\frac{\text{gal}}{\text{hr}},\ V(35.475) = 800.$$

Therefore, we find $V(t) = 800 - 2(t - 35.475)$. The initial value problem for the amount of pollutant in the tank as a function of time is then

$$\frac{dP}{dt} = \overbrace{2\frac{\text{gal}}{\text{hr}} \cdot 0\frac{\text{oz}}{\text{gal}}}^{\text{rate in}} - \overbrace{4\frac{\text{gal}}{\text{hr}} \cdot \frac{P}{V}\frac{\text{oz}}{\text{gal}}}^{\text{rate out}},$$

with the initial condition that $P(35.475) = 500$ ounces. Therefore we must solve the initial value problem

$$\frac{dP}{dt} = -4\frac{P}{800 - 2(t - 35.475)},\ P(35.475) = 500.$$

Again this is a separable equation, and solving it, we obtain

$$P(t) = C(800 - 2(t - 35.475))^2$$

and the initial condition tells us that $C = 500/800$. Thus as our final answer for the amount of pollutant in the tank as a function of time is

$$P(t) = \begin{cases} 4000 - 3998e^{-(3/800)t}, & t \leq 35.475 \\ \frac{500}{800}(800 - 2(t - 35.475))^2, & t > 35.475 \end{cases}$$

Problem 3 *More Fluid Mixing*

Initially, a mass of ten grams of salt is dissolved in a 10 liter tank full of water. Then water containing salt at a concentration of 10 grams per liter trickles in at a rate of two liters per hour. A well-mixed solution trickles out at a rate of 3 liters per hour. Find the concentration (in grams per liter) of the salt in the tank at the time when the tank contains 4 liters.

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Solution 3. We will let S be the amount of salt in the tank as a function of time (in grams), and Q be the concentration of salt in the tank (in grams/liter), and V be the volume of water in the tank (in liters), and t time (in hours). Since the rate in of liquid is different from the rate out, we know that V is not constant. In fact V satisfies the IVP

$$\frac{dV}{dt} = 2 - 3,\ V(0) = 10.$$

Solving this, we find $V(t) = 10 - t$. Next, we set up a differential equation for the amount of salt S in the tank as a function of time. We see that

$$\frac{dS}{dt} = \overbrace{2\frac{\text{ltr}}{\text{hr}} \cdot 10\frac{\text{g}}{\text{ltr}}}^{\text{rate in}} - \overbrace{3\frac{\text{ltr}}{\text{hr}} \cdot \frac{S}{V}\frac{\text{g}}{\text{ltr}}}^{\text{rate out}},$$

with the initial condition othat $S(0) = 10$. Therefore the initial value problem we must solve is

$$\frac{dS}{dt} = 20 - \frac{3}{10-t}S, \quad S(0) = 10.$$

This equation is not separable, but it is linear, so we can solve it with an integrating factor. We calculate

$$\mu(t) = e^{\int \frac{3}{10-t}dt} = e^{-3\ln(10-t)} = (10-t)^{-3}.$$

Multiplying the original differential equation by μ, we get the exact equation

$$(10-t)^{-3}\frac{dS}{dt} = 20(10-t)^{-3} - 3(10-t)^{-4}S.$$

We then move the S-terms over to the left hand side, group, and integrate:

$$\begin{aligned} (10-t)^{-3}S' + 3(10-t)^{-4}S &= 20(10-t)^{-3} \\ ((10-t)^{-3}S)' &= 20(10-t)^{-3} \\ (10-t)^{-3}S &= 10(10-t)^{-2} + C \\ S &= 10(10-t) + C(10-t)^3 \end{aligned}$$

Now the initial condition tells us $C = -9/100$, and therefore

$$S = 10(10-t) - \frac{9}{100}(10-t)^3.$$

The concentration as a function of time is therefore $Q = S/V$, giving us

$$Q = \frac{S}{V} = 10 - \frac{9}{100}(10-t)^2.$$

The tank reaches a volume of 4 liters after exactly 6 hours. The concentration at this time is then seen to be

$$Q(6) = 10 - \frac{9}{100}(10-6)^2 = \frac{214}{25} \approx 8.56 \text{ g per liter.}$$

Problem 4 *Monetary Investment*

A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

(a) Determine the sum $S(t)$ accumulated at any time t

(b) If $r = 7.5\%$ determine k so that 1 million will be available for retirement in 40 years

(c) If $k = 2000$ per year, determine the return rate r that must be obtained to have 1 million available in 40 years

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Solution 4.

(a) Let t be time in years. Then S satisfies the differential equation

$$\frac{dS}{dt} = rS + k,$$

which has the integrating factor $\mu(t) = e^{-rt}$. Using this to solve:

$$\begin{aligned}
-re^{-rt}S + e^{rt}\frac{dS}{dt} &= ke^{-rt} \\
\left(e^{-rt}S\right)' &= ke^{-rt} \\
\int \left(e^{-rt}S\right)' dt &= \int ke^{-rt}dt \\
e^{-rt}S &= -\frac{k}{r}e^{-rt} + C \\
S &= -\frac{k}{r} + Ce^{rt}
\end{aligned}$$

Since there is no initial capital, $S(0) = 0$, and therefore $C = k/r$, making

$$S = \frac{k}{r}\left(e^{rt} - 1\right)$$

(b) Given that $r = 0.075$, we want $S(40) = 10^6$. Solving for k we obtain

$$\begin{aligned}
10^6 &= \frac{k}{0.075}\left(e^{0.075*40} - 1\right) \\
10^6 &= 254.474k \\
k &= 3929.68 \text{ dollars per year}
\end{aligned}$$

(c) Given that $k = 2000$, we want $S(40) = 10^6$. Solving for r we obtain

$$10^6 = \frac{2000}{r}\left(e^{r*40} - 1\right)$$
$$r = 0.097734$$

So we'd need a rate of 9.7734%.

Problem 5 *More Fluid Mixing*

A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos(t))$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

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Solution 5. Notice first of all that more water enters the tank than leaves the tank. In fact, the volume satisfies the differential equation

$$\frac{dV}{dt} = \overbrace{9}^{\text{gal/hr in}} - \overbrace{6}^{\text{gal/hr out}},$$

and therefore $\frac{dV}{dt} = 3$, so that $V = 3t + V_0$, where V_0 is the initial volume ($V_0 = 600$). Thus

$$V = 3t + 600.$$

The weight W of of salt in the tank (in pounds) satisfies the differential equation

$$\frac{dW}{dt} = \text{rate in} - \text{rate out},$$

where

$$\text{rate in} = \overbrace{\frac{1}{5}(1 + \cos(t))}^{\text{lbs salt/gallon in}} \times \overbrace{9}^{\text{gal/hr in}}$$

and

$$\text{rate out} = \overbrace{\frac{W}{V}}^{\text{lbs salt/gallon out}} \times \overbrace{6}^{\text{gal/hr out}}.$$

Thus

$$\frac{dW}{dt} = \frac{9}{5}(1 + \cos(t)) - \frac{2W}{t + 200}$$

This equation is linear! An integrating factor is $\mu(t) = (t+200)^2$. Using this to solve, we get

$$\begin{aligned}
2(t+200)W + (t+200)^2\frac{dW}{dt} &= \frac{9}{5}(1+\cos(t))(t+200)^2 \\
((t+200)^2W)' &= \frac{9}{5}(1+\cos(t))(t+200)^2 \\
\int ((t+200)^2W)'dt &= \int \frac{9}{5}(1+\cos(t))(t+200)^2 dt \\
(t+200)^2W &= \frac{9}{5}(t+200)^2\sin(t) + \frac{18}{5}(t+200)\cos(t) \\
&\quad - \frac{18}{5}\sin(t) + \frac{9}{5}(t+200)^3 + C
\end{aligned}$$

so that

$$W = \frac{9}{5}\sin(t) + \frac{18}{5}\frac{\cos(t)}{t+200} - \frac{18}{5}\frac{\sin(t)}{(t+200)^2} + \frac{9}{5}(t+200) + C$$

Sinc initially there are 5 pounds of salt dissolved in the tank $W(0) = 5$, so that

$$5 = \frac{18}{5}\frac{1}{200} + \frac{9}{5}(200) + C$$

and therefore $C = -355.018$, making

$$W = \frac{9}{5}\sin(t) + \frac{18}{5}\frac{\cos(t)}{t+200} - \frac{18}{5}\frac{\sin(t)}{(t+200)^2} + \frac{9}{5}(t+200) - 355.018$$

Now from our equation for V, we know that the tank overflows at $t = 300$. Evaluating $W(300)$, we obtain

$$W(300) = 543.182 \text{ lbs}$$

Problem 6 Whale Fall

The expression "whale fall" refers to the body of a deceased whale which has fallen to the ocean floor. Suppose that a whale dies of old age (after living a long and happy life, so there's nothing to blubber about). The whale immediately begins to sink, from rest at its initial position on the surface. The whale has mass m and cross-sectional area a.

(i) the density of ocean water is $\rho = 1027$ kg/m^3

(ii) the gravitational acceleration is $g = 9.81$ m/s^2

(iii) the force of drag satisfies the "drag equation" $F_D = \frac{1}{2}\rho u^2 c_d a$ where here a is the cross-sectional area and c_d is the coefficient of drag

(iv) the whale's body descends to the ocean floor, h meters down, unmolested by other life

With this in mind, answer the following questions

(a) set up a first-order initial value problem describing the speed u of the carcass as a function of time

(b) show that $u(t) = K\tanh(gt/K)$ is a solution to this initial value problem, where here $K = \sqrt{2mg/(\rho c_d A)}$ is called the *terminal velocity*, which is the maximum speed of the falling body

(c) find an equation, in terms of K and h, for how long it takes the whale to reach the ocean floor

(d) for a blue whale, we may approximate $a = 28$ meters, $m = 122$ tonnes, and $c_d = 0.75$. If the depth of the ocean is $h = 2$ km, how long will the descent take?

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Solution 6.

(a) We ignore the force of buoyancy in this problem – as a result, our prediction of the time it takes to descend will be smaller than the reality. Without buoyancy, the forces acting on the whale include the force of gravity, and the drag force caused by the water. Newton's law says $F = mu'(t)$, and therefore

$$mu'(t) = mg - \frac{1}{2}\rho u^2 c_d a,$$

which we can simplify by dividing by m on both sides. Moreover the whale is said to begin to sink, so $u(0) = 0$. Thus we have the initial value problem

$$u' = g - \frac{1}{2m}\rho u^2 c_d a.$$

(b) In terms of the constant K, the above initial value problem says

$$u'(t) = g - \frac{g}{K^2}u^2, \quad u(0) = 0.$$

For our value of $u(t)$, we have $u'(t) = g\operatorname{sech}^2(gt/K)$. Moreover, since $1 - \tanh^2(y) = \operatorname{sech}^2(y)$ we see that

$$g - \frac{g}{K^2}u^2 = g(1 - \tanh^2(gt/K)) = g\operatorname{sech}^2(gt/K) = u'(t).$$

Thus $u(t)$ satisfies the differential equation. Moreover, since $\tanh(0) = 0$, we have that $u(0) = 0$, and therefore $u(t)$ satisfies the initial condition. Thus $u(t)$ is a solution to the initial value problem.

(c) Since the whale is initially at the surface, the depth $d(t)$ of the whale as a function of time is given by

$$d(t) = \int_0^t u(t)dt = \int_0^t K\tanh(gt/K) = \frac{K^2}{g}\ln(\cosh(gt/K)).$$

The whale is at the bottom when $d(t) = h$, which gives

$$\frac{K^2}{g}\ln(\cosh(gt/K)) = h.$$

Solving for t, we obtain find that the time t_F it takes to fall is

$$t_F = \frac{K}{g}\text{arccosh}(e^{gh/K^2}).$$

(d) For the numbers given above, $K \approx 1/3$ and therefore $t_F \approx 5994$ seconds, which is around an hour and fourty minutes.

Problem 7 *Jean Wilder's Famous Problem*

A population of Oompa Loompas in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks time. Also on any given day there is a net migration into the area of 15 Oompa Loompas and 16 are eaten by Wangdoodles, Hornswogglers, Snozzwangers and rotten, Vermicious Knids and 7 die of natural causes. If there are initially 100 Oompa Loompas in the area, will the population survive? If not, when do they die out?

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Solution 7. Let n_O be the number of Oompa Loopas in our region, and let t be time in days. Then they satisfy the differential equation

$$\frac{dn_O}{dt} = \text{increase} - \text{decrease}$$

where

$$\text{increase} = \overbrace{rn_O}^{\text{growth due to breeding}} + \overbrace{15}^{\text{growth from migration}}$$

and

$$\text{decrease} = \overbrace{16}^{\text{killed by terrifying monsters}} + \overbrace{7}^{\text{die of natural causes}}.$$

Here r is the growth rate. Thus

$$\frac{dn_O}{dt} = rn_O - 8.$$

An integrating factor for this solution is $\mu(t) = e^{-rt}$. Using this to solve, we get

$$\begin{aligned} -re^{-rt}n_O + e^{-rt}\frac{dn_O}{dt} &= -8e^{-rt} \\ (e^{-rt}n_O)' &= -8e^{-rt} \\ \int (e^{-rt}n_O)'dt &= \int -8e^{-rt}dt \\ e^{-rt}n_O &= \frac{8}{r}e^{-rt} + C \\ n_O &= \frac{8}{r} + Ce^{rt} \end{aligned}$$

Since $n_O(0) = 100$, we know that $C = 100 - \frac{8}{r}$, and therefore

$$n_O = \frac{8}{r} + \left(100 - \frac{8}{r}\right)e^{rt}.$$

What is r, though? Outside external influences (such as birth, death, predator interaction, and migration), the population would satisfy the IVP

$$n_O' = rn_0, \quad n_0(0) = 100,$$

which has the solution $n_O = 100e^{rt}$. From the question, we know that in this case the population should triple in 14 days (2 weeks), so that $n_O(14) = 300$. Thus $300 = 100e^{14r}$, making $r = \ln(3)/14$. Thus the population actually satisfies

$$n_O = \frac{8}{\ln(3)/14} + \left(100 - \frac{8}{\ln(3)/14}\right)e^{rt}.$$

or approximately

$$n_O = 101.947 - 1.947e^{\frac{\ln(3)}{14}t}.$$

This is decreasing, so the population dies out! Setting $n_O = 0$ and solving for t, we find that no more Oompa Loompas are left after about 50.442 days.

Problem 8 *Bernoulli Equations*

A Bernoulli equation is a nonlinear equation of the form

$$y' + p(t)y = q(t)y^n$$

If $n \neq 0$ and $n \neq 1$, then substituting $u = y^{1-n}$ and differentiating yeilds

$$u' = (1-n)y^{-n}y'.$$

This tells us that $y' = \frac{y^n}{(1-n)}u'$. Putting this back into the original differential equation then says

$$\frac{y^n}{1-n}u' + p(t)y = q(t)y^n.$$

Dividing both sides by y, we then get

$$\frac{y^{n-1}}{1-n}u' + p(t) = q(t)y^{n-1}.$$

Now if we notice that $y^{n-1} = 1/u$, then this means

$$\frac{1/u}{1-n}u' + p(t) = q(t)(1/u),$$

which simplifies to

$$\frac{1}{1-n}u' + p(t)u = q(t),$$

which is a linear equation in u. We've just made a nonlinear equation into a linear equation... a small miracle. We can then solve for u, and then use the fact that $u = y^{1/n}$ to obtain y. Let's call this method "Bernoulli's method".

(a) Use Bernoulli's method to solve the differential equation

$$y' = (\Gamma\cos(t) + T)y - y^3$$

where here Γ and T are constants. This equation comes up in the study of stability in fluid flows.

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Solution 8.

(a) We do the substitution $u = y^{-2}$, so that

$$u' = -2y^{-3}y'$$

and

$$y' = -\frac{1}{2}y^3u'$$

Substituting this into the original differential equation for y', we get

$$-\frac{1}{2}y^3u' = (\Gamma\cos(t) + T)y - y^3.$$

Dividing through by y^3 on both sides, this becomes

$$-\frac{1}{2}u' = (\Gamma\cos(t) + T)y^{-2} - 1.$$

Now remembering $u = y^{-2}$:

$$-\frac{1}{2}u' = (\Gamma\cos(t) + T)u - 1.$$

which simplifies to

$$\frac{1}{2}u' + (\Gamma\cos(t) + T)u = 1.$$

This equation is linear! An integrating factor for this equation is

$$\mu(t) = \exp\left(2\Gamma\sin(t) + 2Tt\right),$$

and using this to solve the equation, we get

$$u = \frac{1}{\exp\left(2\Gamma\sin(t) + 2Tt\right)} \int \exp\left(2\Gamma\sin(t) + 2Tt\right) dt$$

so that

$$y = \sqrt{\frac{\exp\left(2\Gamma\sin(t) + 2Tt\right)}{\int \exp\left(2\Gamma\sin(t) + 2Tt\right) dt}}$$