# MATH 307: Problem Set #4

Due on: May 11, 2015

## Problem 1 *Wronskian*

For each of the following collections of functions, show that the collection is linearly independent.

(a) $e^x, xe^x$

(b) $e^x \cos(x)$, $e^x \sin(x)$

(c) $e^{2x}, e^{3x}, e^{5x}$

(d) $1, x, x^2$

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**Solution 1.**

(a) The Wronskian is $e^{2x}$, which is nonzero. Therefore the functions are linearly independent.

(b) The Wronskian is $e^{2x}$, which is nonzero. Therefore the functions are linearly independent.

(c) The Wronskian is $6e^{10x}$, which is nonzero. Therefore the functions are linearly independent.

(d) The Wronskian is 2, which is nonzero. Therefore the functions are linearly independent.

## Problem 2 *Homogeneous ODEs with Const. Coeffs: Distinct Roots*

In each of the following, find the general solution of the given differential equation

(a) $y'' + 3y' + 2y = 0$

(b) $2y'' - 3y' + y = 0$

(c) $y'' - 2y' - 2y = 0$

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**Solution 2.**

(a) The characteristic equation is $r^2 + 3r + 2$, which has roots $r_1 = -1$ and $r_2 = -2$. Hence the general solution is

$$y = C_1 e^{-t} + C_2 e^{-2t}.$$

(b) The characteristic equation is $2r^2 - 3r + 1$, which has roots $r_1 = 1/2$ and $r_2 = 1$. Hence the general solution is

$$y = C_1 e^{t/2} + C_2 e^{t}.$$

(c) The characteristic equation is $r^2 - 2r - 2$, which has roots $r_1 = 1 + \sqrt{3}$ and $r_2 = 1 - \sqrt{3}$. Hence the general solution is

$$y = C_1 e^{(1+\sqrt{3})t} + C_2 e^{(1-\sqrt{3})t}.$$

## Problem 3 *Homogeneous IVPs with Const. Coeffs: Distinct Roots*

In each of the following, find the solution of the IVP

(a) $y'' + 4y' + 3y = 0, \quad y(0) = 2,\ y'(0) = -1$

(b) $y'' + 3y' = 0, \quad y(0) = -2,\ y'(0) = 3$

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**Solution 3.** (a) The characteristic equation is $r^2+4r+3 = 0$ which has roots $r_1 = -1$ and $r_2 = -3$. Hence the general solution is

$$y = C_1 e^{-t} + C_2 e^{-3t}.$$

We calculate then that

$$y' = -C_1 e^{-t} - 3C_2 e^{-3t}.$$

This means that $y(0) = C_1 + C_2$ and $y'(0) = -C_1 - 3C_2$. Therefore our initial condition tells us

$$\begin{aligned} C_1 + C_2 &= 2 \\ -C_1 - 3C_2 &= -1, \end{aligned}$$

and solving this, we find $C_1 = 5/2$ and $C_2 = -1/2$. Therefore the solution is

$$y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

(b) The characteristic equation is $r^2 + 3r = 0$ which has roots $r_1 = 0$ and $r_2 = -3$. Hence the general solution is

$$y = C_1 + C_2 e^{-3t}.$$

We calculate then that

$$y' = -3C_2 e^{-3t}.$$

This means that $y(0) = C_1 + C_2$ and $y'(0) = -3C_2$. Therefore our initial condition tells us

$$\begin{aligned} C_1 + C_2 &= -2 \\ -3C_2 &= 3, \end{aligned}$$

and solving this, we find $C_1 = -1$ and $C_2 = -1$. Therefore the solution is

$$y = -1 - e^{-3t}.$$

## Problem 4 *Complex Number Problems*

In each of the following, rewrite the expression in the form $a + ib$

(a) $e^{2-3i}$

(b) $e^{2-(\pi/2)i}$

(c) $\pi^{-1+2i}$

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**Solution 4.**

(a)

$$\begin{aligned} e^{2-3i} &= e^2 e^{-3i} = e^2(\cos(-3) + i\sin(-3)) \\ &= e^2(\cos(3) - i\sin(3)) = e^2\cos(3) - ie^2\sin(3) \end{aligned}$$

(b)

$$\begin{aligned} e^{2-(\pi/2)i} &= e^2 e^{-(\pi/2)i} = e^2(\cos(-\pi/2) + i\sin(-\pi/2)) \\ &= e^2(0 - i) = -ie^2 \end{aligned}$$

(c)

$$\begin{aligned} \pi^{-1+2i} &= (\pi)^{-1+2i} = (e^{\ln(\pi)})^{-1+2i} = e^{\ln(\pi)(-1+2i)} \\ &= e^{-\ln(\pi)+2\ln(\pi)i} = e^{-\ln(\pi)}e^{2\ln(\pi)i} \\ &= e^{-\ln(\pi)}(\cos(2\ln(\pi)) + i\sin(2\ln(\pi))) \\ &= e^{-\ln(\pi)}\cos(2\ln(\pi)) + ie^{-\ln(\pi)}\sin(2\ln(\pi)) \\ &= \frac{1}{\pi}\cos(2\ln(\pi)) + i\frac{1}{\pi}\sin(2\ln(\pi)) \end{aligned}$$

## Problem 5 *Homogeneous ODEs with Const. Coeffs: Complex Roots*

In each of the following, find the general solution of the ODE

(a) $y'' - 2y' + 6y = 0$

(b) $y'' + 2y' + 2y = 0$

(c) $y'' + 4y' + 6.25y = 0$

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**Solution 5.**

(a) The corresponding characteristic equation is $r^2 - 2r + 6 = 0$, which has roots $1 \pm \sqrt{5}i$. Hence the general solutions is

$$y = C_1 e^t \cos(\sqrt{5}t) + C_2 e^t \sin(\sqrt{5}t).$$

(b) The corresponding characteristic equation is $r^2 + 2r + 2 = 0$, which has roots $r_1 = -1 \pm i$. Hence the general solutions is

$$y = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t).$$

(c) The corresponding characteristic equation is $r^2 + 4r + 6.25 = 0$, which has roots $r_1 = -2 \pm \frac{3}{2}i$. Hence the general solutions is

$$y = C_1 e^{-2t}\cos(3t/2) + C_2 e^{-2t}\sin(3t/2).$$

## Problem 6 *Homogeneous IVPs with Const. Coeffs: Complex Roots*

In each of the following, find the solution of the IVP

(a) $y'' + 4y = 0, \quad y(0) = 0,\ y'(0) = 1$

(b) $y'' + 4y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 0$

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**Solution 6.**

(a) The corresponding characteristic equation is $r^2 + 4 = 0$, which has roots $\pm 2i$. Hence the general solution is

$$y = C_1 \cos(2t) + C_2 \sin(2t).$$

Therefore

$$y' = -2C_1 \sin(2t) + 2C_2 \cos(2t),$$

and it follows that $y(0) = C_1$ and $y'(0) = 2C_2$. Then our initial condition tells us

$$C_1 = 0$$
$$2C_2 = 1$$

and therefore $C_1 = 0$ and $C_2 = 1/2$, so that the solution to the initial value problem is

$$y = \frac{1}{2}\sin(2t).$$

(b) The corresponding characteristic equation is $r^2 + 4r + 5 = 0$, which has roots $-2 \pm i$. Hence the general solution is

$$y = C_1 e^{-2t}\cos(t) + C_2 e^{-2t}\sin(t).$$

Therefore

$$y' = -2C_1 e^{-2t}\cos(t) - C_1 e^{-2t}\sin(t) - 2C_2 e^{-2t}\sin(t) + C_2 e^{2t}\cos(t),$$

and it follows that $y(0) = C_1$ and $y'(0) = -2C_1 + C_2$. Then our initial condition tells us

$$C_1 = 1$$
$$-2C_1 + C_2 = 0$$

and therefore $C_1 = 1$ and $C_2 = 2$, so that the solution to the initial value problem is

$$y = e^{-2t}\cos(t) + 2e^{-2t}\sin(t).$$

## Problem 7 *Homogeneous ODEs with Const. Coeffs: Repeated Roots*

In each of the following, find the general solution of the ODE

(a) $9y'' + 6y' + y = 0$

(b) $4y'' + 12y' + 9y = 0$

(c) $y'' - 6y' + 9y = 0$

(d) $25y'' - 20y' + 4y = 0$

. . . . . . . . .

**Solution 7.**

(a) The roots of the characteristic equation are $r_1 = r_2 = -1/3$, and therefore the general solution is

$$y = C_1 e^{-t/3} + C_2 t e^{-t/3}.$$

(b) The roots of the characteristic equation are $r_1 = r_2 = -3/2$, and therefore the general solution is

$$y = C_1 e^{-3t/2} + C_2 t e^{-3t/2}.$$

(c) The roots of the characteristic equation are $r_1 = r_2 = 3$, and therefore the general solution is

$$y = C_1 e^{3t} + C_2 t e^{3t}.$$

(d) The roots of the characteristic equation are $r_1 = r_2 = 2/5$, and therefore the general solution is

$$y = C_1 e^{2t/5} + C_2 t e^{2t/5}.$$

## Problem 8 *Reduction of Order*

In each of the following, use the method of reduction of order to find a second solution of the ODE

(a) $t^2 y'' + 2ty' - 2y = 0,\ \ t > 0$ (one solution is $y(t) = t$)

(b) $(x - 1)y'' - xy' + y = 0,\ \ x > 1$ (one solution is $y(x) = e^x$)

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**Solution 8.**

(a) We try a solution of the form $y = v(t)t$. Then $y' = v'(t)t + v(t)$ and $y''(t) = v''(t)t + 2v'(t)$, so that

$$\begin{aligned} t^2y'' + 2ty' - 2y &= t^2(v''(t)t + 2v'(t)) + 2t(v'(t)t + v(t)) - 2(v(t)t) \\ &= t^3v''(t) + 4t^2v'(t). \end{aligned}$$

Then since $t^2y'' + 2ty' - 2y = 0$ (in order to be a solution to the equation), we must have

$$t^3v''(t) + 4t^2v'(t) = 0.$$

Dividing both sides by $t^2$, this simplifies to

$$tv''(t) + 4v'(t) = 0.$$

Now if we substitute $w = v'$, then this equation becomes

$$tw'(t) + 4w(t) = 0.$$

This equation is separable, and the solution is $w(t) = C_1t^{-4}$, where $C_1$ is an arbitrary constant. Then since $v'(t) = w$, it follows that $v(t) = C_1t^{-3} + C_2$ (where we've left $-C_1/3$ as $C_1$ since it's an arbitrary constant anyway). Hence another solution is

$$y = v(t)t = C_1t^{-2} + C_2t,$$

and in fact this is the general solution.

(b) We try a solution of the form $y = v(x)e^x$. Then $y' = v'(x)e^x + v(x)e^x$ and $y'' = v''(x)e^x + 2v'(x)e^x + v(x)e^x$, so that

$$\begin{aligned} &(x-1)y'' - xy' + y \\ &= (x-1)(v''(x)e^x + 2v'(x)e^x + v(x)e^x) - x(v'(x)e^x + v(x)e^x) + (v(x)e^x) \\ &= (x-1)e^xv''(x) + (x-2)e^xv'(x). \end{aligned}$$

Then since $(x-1)y'' - xy' + y = 0$ (in order to be a solution to the equation), we must have

$$(x-1)e^xv''(x) + (x-2)e^xv'(x) = 0.$$

Dividing both sides by $e^x$, this simplifies to

$$(x-1)v''(x) + (x-2)v'(x) = 0.$$

Now if we substitute $w = v'$, the equation becomes

$$(x-1)w'(x) + (x-2)w(x) = 0,$$

which is separable. The solution is

$$w = C_1(x-1)e^{-x}.$$

Then since $v' = w$, it follows that

$$v = -C_1xe^{-x} + C_2.$$

Hence another solution to the original differential equation is

$$y = ve^x = -C_1x + C_2e^x,$$

and in fact this is the general solution.

## Problem 9 *Euler-Cauchy Equation*

A second-order Euler-Cauchy equation is a second-order homogeneous linear ordinary differential equation with non-constant coefficients of the form

$$at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0, \tag{1}$$

where $a, b, c$ are constants with $a \neq 0$. Due to it's regular form, the Euler-Cauchy equation may be transformed into a homogeneous linear ordinary differential equation with constant coefficients, by means of an appropriate variable substitution.

Consider the variable substitution $t = e^u$

(a) Show that

$$\frac{dy}{du} = t\frac{dy}{dt}$$

(b) Show that

$$\frac{d^2y}{du^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

(c) Using (a) and (b), show that the Euler-Cauchy Equation (1) is equivalent to the second-order linear ordinary differential equation with constant coefficients

$$a\frac{d^2y}{du^2} + (b-a)\frac{dy}{du} + cy = 0.$$

. . . . . . . . .

**Solution 9.**

(a) The chain rule tells us that

$$\frac{dy}{du} = \frac{dy}{dt}\frac{dt}{du} = \frac{dy}{dt}e^u = \frac{dy}{dt}t.$$

(b) The previous calculation actually shows that for any function $f$, we have $\frac{d}{du}(f) = t\frac{d}{dt}(f)$. Then since

$$\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{d}{du}y\right)$$

we can replace any occurence of $\frac{d}{du}$ with $t\frac{d}{dt}$. Doing so, we obtain

$$\frac{d^2y}{du^2} = t\frac{d}{dt}\left(t\frac{d}{dt}y\right).$$

Now to simplify this, we need to use the product rule. We find

$$t\frac{d}{dt}\left(t\frac{d}{dt}y\right) = t^2\frac{d^2}{dt^2}y + t\frac{d}{dt}y = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}.$$

Therefore

$$\frac{d^2y}{du^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}.$$

(c) Using (b), we can replace $t^2 \frac{d^2y}{dt^2}$ with $\frac{d^2y}{du^2} - t\frac{dy}{du}$ in Equation (1). Doing so, Equation (1) becomes

$$a\left(\frac{d^2y}{du^2} - t\frac{dy}{du}\right) + bt\frac{dy}{dt} + cy = 0$$

This simplifies to

$$a\frac{d^2y}{du^2} + (b-a)t\frac{dy}{dt} + cy = 0.$$

Now using (a), we can replace $t\frac{dy}{dt}$ with $\frac{dy}{du}$, obtaining

$$a\frac{d^2y}{du^2} + (b-a)\frac{dy}{du} + cy = 0.$$

## Problem 10 *Euler-Cauchy Equation Practice*

Find the general solution to each of the following equations

(a) $t^2y'' + 4ty' + 2y = 0, \;\; t > 0$

(b) $3t^2y'' + 7ty' - 4y = 0, \;\; t > 0$

. . . . . . . . .

**Solution 10.**

(a) Using the previous problem, substituting $t = e^u$ the equation becomes

$$y''(u) + 3y'(u) + 2y(u) = 0.$$

The general solution of this equation is

$$y(u) = Ae^{2u} + Be^{u}.$$

Substituting back in $u = \ln(t)$, we see

$$y(t) = At^2 + Bt.$$

(b) Using the previous problem, substituting $t = e^u$ the equation becomes

$$3y''(u) + 4y'(u) - 4y(u) = 0.$$

The general solution of this equation is

$$y = Ae^{-2x} + Be^{2x/3}.$$

Substituting back in $u = \ln(t)$, we see

$$y(t) = At^{-2} + Bt^{2/3}.$$

## Problem 11 *Higher-Order ODE's*

In this class, we will mostly stick with first and second-order equations. However, it is important to recognizer that many of the methods we outline for first and second order equations naturally generalize to the case of higher-order equations. For each of the following equations, do your best to extend a method we have learned previously, in order to find the general solution.

(a) $y'''' + y = 0$

(b) $y''' - 3y'' - 3y' + y = 0$

(c) $y''' - y'' - y' + y = 0$

(d) $t^3y''' + 3t^2y'' + ty' + y = 0$

. . . . . . . . .

**Solution 11.**

The characteristic polynomial is $r^2 + 1$, which has roots $-\sqrt{2}/2 \pm i\sqrt{2}/2$ and $\sqrt{2}2 \pm i\sqrt{2}/2$. The general solution is then

$$y = Ae^{-\sqrt{2}t/2}\cos(\sqrt{2}t/2)+Be^{-\sqrt{2}t/2}\sin(\sqrt{2}t/2)+Ce^{\sqrt{2}t/2}\cos(\sqrt{2}t/2)+De^{\sqrt{2}t/2}\sin(\sqrt{2}t/2)$$

The characteristic polynomial is $r^3 - 3r^2 - 3r + 1$, which has roots $-1$ and $2 \pm \sqrt{3}$. The general solution is therefore

$$y = Ae^{-t} + Be^{(2+\sqrt{3})t} + Ce^{(2-\sqrt{3})t}.$$

The characteristic polynomial is $r^3 - r^2 - r + 1$, which as roots $1, 1, -1$. The general solution is therefore

$$y = (At + B)e^t + Ce^{-t}.$$

The equation is an Euler-Cauchy equation! We do the substitution $t = e^u$, noting that

$$t^3\frac{d^3y}{dt^3} = \frac{d^3y}{du^3} - 3\frac{d^3y}{du^3} + 2\frac{dy}{du},$$

and remembering that

$$t^2\frac{d^2y}{dt^3} = \frac{d^2y}{du^2} - \frac{dy}{du}$$

$$t\frac{dy}{dt} = \frac{dy}{du}.$$

Therefore the Euler-Cauchy equation becomes

$$y'''(u) + y(u) = 0.$$

The characteristic polynomial of this equation is $r^3 + 1$, and the general solution is therefore

$$y(u) = Ae^{-u} + Be^{u/2}\cos(\sqrt{3}u/2) + Ce^{u/2}\sin(\sqrt{3}u/2).$$

Substituting back in for $u$, we then obtain

$$y(t) = At^{-1} + B\sqrt{t}\cos(\sqrt{3}\ln(t)/2) + C\sqrt{t}\sin(\sqrt{3}\ln(t)/2).$$