MATH 307: Problem Set #5

Due on: February 19, 2016

Problem 1 *Method of Undetermined Coefficients: General Solutions*

In each of the following, find the general solution of the given differential equation

(a) $y'' - 2y' - 3y = 3e^{2t}$

(b) $y'' - 2y' - 3y = -3te^{-t}$

(c) $y'' - 2y' - 3y = te^{-t} + 7e^{2t}$

(d) $y'' - 2y' - 3y = 2te^{-t} - 3e^{2t}$

(e) $y'' - 2y' - 3y = 4te^{-t} + e^{2t}$

(f) $y'' + 2y' + 5y = \sin(2t)$

(g) $y'' + 2y' + 5y = \cos(2t)$

(h) $y'' + 2y' + 5y = 4\sin(2t) + 7\cos(2t)$

(i) $y'' + 2y' = 3 + 4\sin(2t)$

(j) $y'' + 2y' + y = 2e^{-t}$

(k) $y'' + y = 3\sin(2t)$

(l) $y'' + y = t\cos(2t)$

(m) $y'' + y = 3\sin(2t) + t\cos(2t)$

(n) $y'' - y' - 2y = e^{t}$

(o) $y'' - y' - 2y = e^{-t}$

(p) $y'' - y' - 2y = \cosh(t)$ [Hint: $\cosh(t) = (e^{t} + e^{-t})/2$]

. . . .

Solution 1.

(a) The general solution of the corresponding homogeneous equation is

$$y_h = C_1e^{3t} + C_2e^{-t}.$$

Therefore, we must try a particullar solution of the form $y_p = Ae^{2t}$. Putting this into the differential equation, we find that $A = -1$. From this, we see that the general solution of the equation is

$$y = y_h + y_p = C_1e^{3t} + C_2e^{-t} - e^{2t}.$$

(b) The general solution of the corresponding homogeneous equation is

$$y_h = C_1e^{3t} + C_2e^{-t}.$$

Therefore, we must try a particullar solution of the form $y_p = (At^2 + Bt)e^{-t}$. Putting this into the differential equation, we find that $A = 3/8$ and $B = 3/16$. From this, we see that the general solution of the equation is

$$y = y_h + y_p = C_1e^{3t} + C_2e^{-t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}.$$

(c) Let y_1 be the particular solution found in part (a) and y_2 be the particular solution found in part (b), and let L be the linear differential operator

$$L\{y\} = y'' - 2y' - 3y.$$

Then the equation that we are trying to solve is

$$L\{y\} = te^{-t} + 7e^{2t}$$

Since L is a linear operator and $L\{y_1\} = 3e^{2t}$ and $L\{y_2\} = -3te^{-t}$, for any constants A and B we have that

$$L\{Ay_1 + By_2\} = AL\{y_1\} + BL\{y_2\} = 3Ae^{2t} + -3Bte^{-t}.$$

Thus if we choose $A = 7/3$ and $B = -1/3$, then we get a particular solution for our differential equation

$$y_p = \frac{7}{3}y_1 - \frac{1}{3}y_2 = -\frac{7}{3}e^{2t} + \left(\frac{-1}{8}t^2 + \frac{-1}{16}t\right)e^{-t}$$

Thus the general solution is

$$y = C_1e^{3t} + C_2e^{-t} - \frac{7}{3}e^{2t} + \left(\frac{-1}{8}t^2 + \frac{-1}{16}t\right)e^{-t}.$$

(d) Using a similar argument as in part (c), we get the general solution

$$y = C_1e^{3t} + C_2e^{-t} + e^{2t} + \left(\frac{-2}{8}t^2 + \frac{-2}{16}t\right)e^{-t}.$$

(e) Using a similar argument as in part (c), we get the general solution

$$y = C_1e^{3t} + C_2e^{-t} - \frac{1}{3}e^{2t} + \left(\frac{-4}{8}t^2 + \frac{-4}{16}t\right)e^{-t}.$$

(f) The general solution of the corresponding homogeneous equation is

$$y_h = C_1e^{-t}\cos(2t) + C_2e^{-t}\sin(2t)$$

To get a particular solution, we instead find a particular solution to the complex equation

$$\widetilde{y}'' + 2\widetilde{y}' + 5\widetilde{y} = e^{2it}.$$

To solve this equation, we try a solution of the form $\widetilde{y}_p = Ae^{2it}$. Putting this into the differential equationn, we find that

$$A = \frac{1}{1+4i} = \frac{1-4i}{17} = \frac{1}{17} - \frac{4}{17}i,$$

and therefore

$$\begin{aligned}\widetilde{y}_p &= \left(\frac{1}{17} - \frac{4}{17}i\right)e^{2t} \\ &= \left(\frac{1}{17} - \frac{4}{17}i\right)(\cos(2t) + i\sin(2t)) \\ &= \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t) - \frac{4}{17}i\cos(2t) + \frac{1}{17}i\sin(2t)\end{aligned}$$

This means that

$$y_p = \operatorname{Im}\{\widetilde{y}_p\} = -\frac{4}{17}\cos(2t) + \frac{1}{17}\sin(2t)$$

Thus the general solution is

$$y = y_h + y_p = C_1e^{-t}\cos(2t) + C_2e^{-t}\sin(2t) - \frac{4}{17}\cos(2t) + \frac{1}{17}\sin(2t)$$

(g) The general solution to the corresponding homogeneous equation is the same as in (f). To get a particular solution, we instead find a particular solution to the complex equation

$$\widetilde{y}'' + 2\widetilde{y}' + 5\widetilde{y} = e^{2it}.$$

we solved this in part (f) and found

$$\widetilde{y}_p = \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t) - \frac{4}{17}i\cos(2t) + \frac{1}{17}i\sin(2t)$$

Therefore

$$y_p = \operatorname{Re}\{\widetilde{y}_p\} = \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t)$$

so that the general solution is

$$y = y_h + y_p = C_1e^{-t}\cos(2t) + C_2e^{-t}\sin(2t) + \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t)$$

(h) The general solution to the corresponding homogeneous equation is the same as in (f) and (g). The particular solution of the equation will be a linear combination of the particular solutions found in (f) and (g). In particular, if y_1 is the particular solution found in (f) and y_2 is the particular solution found in (g), then the particular solution to the differential equation in (h) will be

$$y_p = 4y_1 + 7y_2 = \frac{-9}{17}\cos(2t) + \frac{32}{17}\sin(2t)$$

Thus the general solution is

$$y = y_h + y_p = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) + \frac{-9}{17}\cos(2t) + \frac{32}{17}\sin(2t).$$

(i) The general solution to the corresponding homogeneous equation is

$$y_h = C_1 + C_2 e^{-2t}.$$

To find a particular solution, we will find a particular solution of the equation

$$y_1'' + 2y_1' = 3$$

and a particular solution of the equation

$$y_2'' + 2y_2' = 4\sin(2t)$$

and then add them together: $y_p = y_1 + y_2$. To find y_1, notice that 3 is a solution of the homogeneous equation, so we should try $y_1 = At$. Doing so, we find $A = 3/2$, so that $y_1 = 3t/2$. Then to find y_2, we solve the corresponding complexified equation, like we did in problem (f). Doing so, we find $y_2 = -\frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t)$. Therefore the general solution is

$$y = y_h + y_p = y_h + y_1 + y_2 = C_1 + C_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t).$$

(j) The general solution to the corresponding homogeneous equation is

$$y_h = C_1 e^{-t} + C_2 t e^{-t}.$$

Since e^{-t} corresponds to a double root of the characteristic polynomial of this equation, to get a particular solution we should try $y_p(t) = At^2 e^{-t}$. Doing so, we find $A = 1$, and therefore the general solution is

$$y_h = C_1 e^{-t} + C_2 t e^{-t} + t^2 e^{-t}.$$

(k) The general solution of the corresponding homogeneous equation is

$$y_h = C_1\cos(t) + C_2\sin(t).$$

By complexifying and taking the imaginary component, we also find a particular solution of the form $y_p = -\sin(2t)$. Therefore the general solution is

$$y = y_h + y_p = C_1\cos(t) + C_2\sin(t) - \sin(2t).$$

(l) The general solution of the corresponding homogeneous equation is

$$y_h = C_1 \cos(t) + C_2 \sin(t).$$

To find a particular solution, we instead find a particular solution of the equation

$$\widetilde{y}'' + \widetilde{y} = te^{2it}$$

and take the real component. Doing so, we find a particular solution of the form $y_p = \frac{4}{9}\sin(2t) - \frac{1}{3}t\cos(2t)$. Therefore the general solution is

$$y = y_h + y_p = C_1 \cos(t) + C_2 \sin(t) + \frac{4}{9}\sin(2t) - \frac{1}{3}t\cos(2t).$$

(m) Let y_1 be the particular solution to part k and y_2 be the particular solution to part (l), then the particular solution here will be $y_p = y_1 + y_2$, so that

$$y_p = -\frac{5}{9}\sin(2t) - \frac{1}{3}t\cos(2t)$$

The general solution is therefore

$$y = y_h + y_p = C_1 \cos(t) + C_2 \sin(t) - \frac{5}{9}\sin(2t) - \frac{1}{3}t\cos(2t).$$

(n) The general solution of the corresponding homogeneous equation is

$$y_h = C_1 e^{2t} + C_2 e^{-t}.$$

To find a particular solution, we try $y_p = Ae^t$. Putting this into the differential equation, we find $A = -1/2$, so that the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{2}e^t.$$

(o) The general solution to the homogeneous equation is the same as in part (n). Since -1 is a root of the characteristic polynomial, to find the particular solution we try a solution of the form $y_p = Ate^{-t}$. Putting this into the differential equation, we find $A = 1/3$, so that the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{3}te^{-t}.$$

(p) The general solution to the homogeneous equation is the same as in part (n). If y_1 is the particular solution to part (n) and y_2 is the particular solution to part (o), then the particular solution here will be $\frac{1}{2}y_1 + \frac{1}{2}y_2$. Therefore the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{4}e^t - \frac{1}{6}te^{-t}.$$

Problem 2 *Method of Undetermined Coefficients: Initial Value Problems*

In each of the following, find the solution of the given initial value problem

(a) $y'' + 4y = t^2 + 3e^t$, $y(0) = 0$, $y'(0) = 2$

(b) $y'' - 2y' - 3y = 3te^{2t}$, $y(0) = 1$, $y'(0) = 0$

(c) $y'' + 2y' + 5y = 4e^{-t}\cos(2t)$, $y(0) = 1$, $y'(0) = 0$

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Solution 2.

(a) $y(t) = \frac{7}{10}\sin(2t) - \frac{19}{40}\cos(2t) - \frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t$

(b) $y(t) = e^{3t} + \frac{2}{3}e^{-t} + \left(\frac{-2}{3} - t\right)e^{2t}$

(c) $y(t) = e^{-t}\cos(2t) + \left(t + \frac{1}{2}\right)e^{-t}\sin(2t)$