MATH 307: Problem Set #6

Due on: Feb 26, 2016

Problem 1 *Trigonometric Forcing*

Find a particular solution to each of the following differential equations

(a) $y'' + 2y' + y = \sin(t)$

(b) $y'' + 2y' + y = \cos(t)$

(c) $y'' + 2y' + y = 3\sin(t) + 2\cos(t)$

(d) $y'' + y = \cos(t)$

(e) $y'' + y = e^{-2t}\sin(t)$

(f) $y'' + 2y' + 2y = e^{-t}\cos(t)$

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Solution 1.

(a) $y_p = -\frac{1}{2}\cos(t)$

(b) $y_p = \frac{1}{2}\sin(t)$

(c) $y_p = \frac{-3}{2}\cos(t) + \sin(t)$

(d) $y_p = \frac{1}{2}t\sin(t)$

(e) $y_p = \frac{1}{8}e^{-2t}\sin(t) + \frac{1}{8}e^{-2t}\cos(t)$

(f) $y_p = \frac{1}{2}te^{-t}\sin(t)$

Problem 2 *Trigonometry Exercise*

In each of the following, determine ω_0, R, δ so as to write the given expression in the form $u = R\cos(\omega_0 t - \Delta)$.

(a) $u = 3\cos(2t) + 4\sin(2t)$

(b) $u = 4\cos(3t) - 2\sin(3t)$

(c) $u = -2\cos(\pi t) - 3\sin(\pi t)$

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Solution 2.

(a) $R = 5$, $\omega_0 = 2$, $\delta = \tan^{-1}(4/3) = 0.9273$

(b) $R = 2\sqrt{5}$, $\omega_0 = 3$, $\delta = \tan^{-1}(-2/4) = -0.4636$

(c) $R = \sqrt{13}$, $\omega_0 = \pi$, $\delta = \tan^{-1}(-3/-2) + \pi = -2.1588$

Problem 3 *A Spring Problem*

A mass weighing 3 lbs stretches a spring 3 inches. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of motion.

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Solution 3. We calculate $k = 3/3 = 1$ lb/in, $\gamma = 0$, $m = 3/32$ slugs. Also $u(0) = -1$ in, $u'(0) = 24$ in. Therefore we have the IVP

$$(3/32)u'' + u = 0,\ \ u(0) = -1,\ \ u'(0) = 24.$$

The solution of this IVP is

$$u(t) = -\cos(\sqrt{(32/3)x}) + 3\sqrt{6}\sin(\sqrt{(32/3)x}).$$

From this $R = \sqrt{55}$, $\omega = 32/3$, $T = 2\pi/\omega = 6\pi/32$, and $\delta = \tan^{-1}(-3\sqrt{6}) + \pi = 1.7060$.

Problem 4 *Another Spring Problem*

A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position u at any time t. Find the quasifrequency μ and the ratio of μ to the natural frequency corresponding to undamped motion.

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Solution 4. We calculate $k = 3/.1 = 30$ N/m, $\gamma = 3/5$ Ns/m, $m = 2$ kg. Also $u(0) = 0.05$ m and $u'(0) = 0.10$ m, so we have the IVP

$$30u'' + (3/5)u' + 2u = 0, \; u(0) = 0.05, \; u'(0) = 0.10$$

The solution of this IVP is

$$u(t) = \frac{1997}{39940} e^{-t/100} \cos(\sqrt{(1997/3)}x/100) + \frac{201\sqrt{5991}}{39940} e^{-t/100} \sin(\sqrt{(1997/3)}x/100).$$

From this $R = 0.3927$, $\omega = 0.2580$, $T = 2\pi/\omega = 24.3529$, and $\delta = 1.4431$.

Problem 5 *LCR Circuit Problem*

If a series circuit has a capacitor of $C = 0.8 \times 10^{-6}$ F and an inductor of $L = 0.2$ H, find the smallest value of the resistance R so that the circuit is critically damped.

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Solution 5. The differential equation is

$$LI'' + RI' + (1/C)I = 0.$$

This is critically damped if $R^2 = 4L/C = 10^6$, eg. if $R = 10^3\ \Omega$. (Here Ω is a unit of measure – read as Ohms.)

Problem 6 *A Forced Spring Problem*

A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.

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Solution 6. We calculate $k = 5 \cdot (9.81)/0.10 = 490.5$ N/m, $\gamma = 2/0.04 = 50$ Ns/m, $m = 5$. Also $u(0) = 0$ m and $u'(0) = 0.03$ m/s, and therefore our initial value problem is

$$5u'' + 50u' + 490.5u = 10\sin(t/2), \; u(0) = 0, \; u'(0) = 0.03.$$

Problem 7 *Another Forced Spring Problem*

If an undamped spring-mass system with a mass that weighs 6 lb and a spring constant of 41 lb/in is suddenly set in motion at $t = 0$ by an external force of $4\cos(7t)$ lb, determine the position of the mass at any time and draw a graph of the displacement versus t.

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Solution 7. The mass is $m = 6/32 = 3/16$ slugs. The initial value problem is

$$(3/16)u'' + 41u = 4\cos(7t),\ u(0) = 0,\ u'(0) = 0.$$

The solution of this IVP is

$$u(t) = (64/509)\cos(7t) - (64/509)\cos(\sqrt{(656/3)}t).$$

(graph excluded)

Problem 8 *A Third Forced Spring Problem*

A mass that weighs 8 lb stretches a spring 6 inches. The system is acted on by an external force of $8\sin(8t)$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

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Solution 8. We calculate $m = 8/32 = 1/4$ slugs, $k = 8/.5$ lb/ft, $\gamma = 0$. Moreover $u(0) = 0.25$ ft, $u'(0) = 0$. Therefore

$$(1/4)u'' + 16u = 8\sin(8t),\ u(0) = 0.25,\ u'(0) = 0.$$

The solution is

$$u = 2t\sin(8t) + 0.25\cos(8t).$$

This is equal to zero when $8t\tan(8t) = -1$. The first four solutions of this are $t = 0.3498, 0.7652, 1.1647$, and 1.5608.

Problem 9 *Continuity Problem*

In each of the following sketch a graph of the function and determine whether it is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

(a)

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3 - t, & 2 < t \le 3 \end{cases}$$

(b)

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3 - t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$$

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Solution 9.

(a) (graph excluded) continuous

(b) (graph excluded) piecewise continuous