MATH 307: Problem Set #7

Due on: Feb 11, 2016

Problem 1 *First-order Variation of Parameters*

The method of variation of parameters uses the homogeneous solutions of a linear ordinary differential equation to determine the nonhomogeneous solutions. In this problem, we explain the method of variation of parameters for first-order linear equations.

Consider the first-order linear equation

$$y' + b(x)y = f(x).$$

The associated homogeneous linear equation is

$$y_h' + b(x)y_h = 0.$$

Note that this equation is separable – and therefore has the solution

$$y_h = C\exp(-\int b(x)dx).$$

The method of variation of parameters is to propose a solution to the nonhomogeneous equation $y = v(x)y_h(x)$, where here $v(x)$ is the "varying parameter of y_h. From this, we calculate

$$y' = v'(x)y_h(x) + v(x)y_h'(x).$$

Furthermore, since $y_h'(x) = -b(x)y_h(x)$, this simplifies to

$$y' = (v'(x) - b(x))y_h(x).$$

Substituting this into the original differential equation yeilds

$$(v'(x) - b(x))y_h(x) + b(x)y_h(x) = f(x).$$

This simplifies to

$$v'(x) = \frac{f(x)}{y_h(x)}.$$

Therefore

$$v(x) = \int \frac{f(x)}{y_h(x)}dx,$$

and since $y = v(x)y_h(x)$, the solution is

$$y(x) = y_h(x) \int \frac{f(x)}{y_h(x)} dx.$$

In summary, the method of variation of parameters is the following.
METHOD OF VARIATION OF PARAMETERS

Step 1: Calculate the solution $y_h(x)$ of the associated homogeneous equation

Step 2: Calculate the integral $\int \frac{f(x)}{y_h(x)} dx$

Step 3: Multiply the integral by $y_h(x)$ to get y

Use the method of variation of parameters to determine the solution of the following initial value problem

$$y' - y/x = x\sin(x).$$

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Problem 2 *Second-order Variation of Parameters*

The method of variation of parameters outlined in the previous problem extends to differential equations of higher order. The idea is still fundamentally the same – we use the homogeneous solutions of a linear ordinary differential equation to determine the nonhomogeneous solutions. In this problem, we explain the method of variation of parameters for second-order linear equations.

Consider the second-order linear equation

$$y'' + b(x)y' + c(x)y = f(x).$$

The associated homogeneous linear equation is

$$y_h'' + b(x)y_h' + c(x)y_h = 0.$$

This is a second-order homogeneous linear ordinary differential equation, and therefore should have two linearly independent solutions $y_1(x)$ and $y_2(x)$.

The method of variation of parameters is to propose a solution to the nonhomogeneous equation

$$y = v_1(x)y_1(x) + v_2(x)y_2(x),$$

satisfying the additional condition that

$$y' = v_1(x)y_1'(x) + v_2(x)y_2'(x).$$

where here $v_1(x)$ and $v_2(x)$ are the "varying parameters" of the homogeneous solutions y_1 and y_2. Then performing a calculation that is similar to, but more involved than the one in the previous problem, we find

$$v_1(x) = \int \frac{-y_2(x)f(x)}{W(x)} dx, \quad v_2(x) = \int \frac{y_1(x)f(x)}{W(x)} dx$$

where here $W(x) := W[y_1; y_2] = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ is the Wronskian of $y_1(x)$ and $y_2(x)$. Then since $y = v_1(x)y_1(x) + v_2(x)y_2(x)$, this shows that

$$y = y_1(x)\int \frac{-y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx.$$

In summary, the method of variation of parameters is the following.
METHOD OF VARIATION OF PARAMETERS

Step 1: Calculate two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the associated homogeneous equation

Step 2: Calculate the integrals

$$v_1(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx, \quad v_2(x) = \int \frac{y_1(x)f(x)}{W(x)}dx$$

Step 3: Multiply the integrals by $y_1(x)$ and $y_2(x)$, respectively, to get y

Use the method of variation of parameters to determine the solution of the following initial value problem

$$y'' + y = 1 + \tan(x).$$

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Problem 3 *Continuity Problem*

In each of the following sketch a graph of the function and determine whether it is continuous, piecewise continuous, or neither on the interval $0 \leq t \leq 3$.

(a)

$$f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 1, & 1 < t \leq 2 \\ 3-t, & 2 < t \leq 3 \end{cases}$$

(b)

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 3-t, & 1 < t \leq 2 \\ 1, & 2 < t \leq 3 \end{cases}$$

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Problem 4 *Laplace Transforms*

In each of the following, deptermine the Laplace transform of the given function $f(t)$. Note that n is a positive integer and a is a real constant.

(a) $f(t) = \cosh(at)$ [Recall that $\cosh(at) = (e^{at} + e^{-at})/2$]

(b) $f(t) = te^{at}$

(c) $f(t) = t\sin(at)$

(d) $f(t) = t^n e^{at}$

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Problem 5 *Inverse Laplace Transforms*

In each of the following, find the inverse Laplace transform of the given function

(a) $F(s) = \frac{3}{s^2+4}$

(b) $F(s) = \frac{4}{(s-1)^3}$

(c) $F(s) = \frac{2}{s^2+3s-4}$

(d) $F(s) = \frac{3s}{s^2-s-6}$

(e) $F(s) = \frac{2s+1}{s^2-2s+2}$

(f) $F(s) = \frac{8s^2-4s+12}{s(s^2+4)}$

(g) $F(s) = \frac{1-2s}{s^2+4s+5}$

(h) $F(s) = \frac{2s-3}{s^2+2s+10}$

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Problem 6 *Solving IVPs*

In each of the following, use Laplace transforms to solve the initial value problem

(a) $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$

(b) $y'' - 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$

(c) $y'' + 2y' + 5y = 0$, $y(0) = 2$, $y'(0) = -1$

(d) $y'' + \omega^2 y = \cos(2t)$, $y(0) = 1$, $y'(0) = 0$ (Here, ω is a constant and $\omega^2 \neq 4$.)

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Problem 7 *Step Functions*

In each of the following, sketch a graph of the given function on the interval $t \geq 0$

(a) $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

(b) $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

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Problem 8 *Brackets to Step Functions*

In each of the following, sketch a graph of the given function on the interval $t \geq 0$, and then convert the function from bracket form to step function form

(a)

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 2 \\ e^{-(t-2)} & \text{if } t \geq 2 \end{cases}$$

(b)

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ t-1 & \text{if } 1 \leq t < 2 \\ t-2 & \text{if } 2 \leq t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

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Problem 9 *Laplace Transforms of Step Functions*

Find the Laplace transform of the given function

(a)

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 2 \\ (t-2)^2 & \text{if } t \geq 2 \end{cases}$$

(b)

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ t^2 - 2t + 2 & \text{if } t \geq 1 \end{cases}$$

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Problem 10 *More Inverse Laplace Transforms*

Find the Inverse Laplace transform of the given function

(a) $F(s) = \frac{e^{-2s}}{s^2+s-2}$

(b) $F(s) = \frac{2e^{-2s}}{s^2-4}$

(c) $F(s) = \frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$

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Problem 11 *Solutions to IVPs with Discontinuous Forcing*

In each of the following problems, find the solution of the given initial value problem

(a) $y'' + 2y' + 2y = h(t)$, $y(0) = 0$, $y'(0) = 1$, $h(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ 1 & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi \le t \end{cases}$

(b) $y'' + 3y' + 2y = f(t)$, $y(0) = 0$, $y'(0) = 0$, $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 10 \\ 0 & \text{if } t \ge 10 \end{cases}$

(c) $y'' + y = u_{3\pi}(t)$, $y(0) = 1$, $y'(0) = 0$.

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