Math 307 Quiz 2

January 24, 2016

Problem 1. Check if each of the following equations is exact. If it is, solve it.

(a)

$$(x + 2y) + (2x - 3\sin(y))y' = 0.$$

(b)

$$x^2 + 2xy = (x^2 + 3y)y'.$$

Solution 1.

(a) We have that

$$M(x, y) = x + 2y, \quad N(x, y) = (2x - 3\sin(y)),$$

and therefore

$$M_y(x, y) = 2, \quad N_x(x, y) = 2.$$

Thus the equation is exact. To solve it, we search for a function $\psi(x, y)$ satisfying

$$\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y).$$

This first equation says that

$$\begin{aligned}\psi(x, y) &= \int \psi_x(x, y)\partial x = \int M(x, y)\partial x \\ &= \int (x + 2y)\partial x = \frac{1}{2}x^2 + 2xy + h(y).\end{aligned}$$

It follows that

$$\psi_y(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + 2xy + h(y)\right) = 2x + h'(y).$$

Then since $\psi_y(x,y) = N(x,y)$, we see that

$$2x + h'(y) = 2x - 3\sin(y).$$

Therefore

$$h'(y) = -3\sin(y),$$

and this means that

$$h(y) = 3\cos(y) + C.$$

Consequently

$$\psi(x,y) = \frac{1}{2}x^2 + 2xy + 3\cos(y) + C,$$

and the solution we get by setting $\psi(x,y)$ equal to a constant B, eg.

$$\frac{1}{2}x^2 + 2xy + 3\cos(y) + C = B.$$

Combining arbitrary coefficients, we can rewrite this as

$$\frac{1}{2}x^2 + 2xy + 3\cos(y) = A.$$

(b) We rewrite the equation in M, N-form:

$$(x^2 + 3y)y' - (x^2 + 2xy) = 0.$$

Therefore we have that

$$M(x,y) = -(x^2 + 2xy), \quad N(x,y) = x^2 + 3y,$$

and we calculate

$$M_y(x,y) = -2x, \quad N_x(x,y) = 2x.$$

These are not equal, so the equation is not exact.

Problem 2. Find two different solutions to the initial value problem

$$y' = y^{1/3}, \quad y(1) = 0.$$

Solution 2. This equation is separable. Separating and integrating, we get

$$\int y^{-1/3} dy = \int 1 dx.$$

Calculating the integrals, this says

$$\frac{3}{2} y^{2/3} = x + C.$$

Solving for y, we get (taking $B = (2/3)C$)

$$y = \pm \left(\frac{2}{3} x + B \right)^{3/2}.$$

The initial condition that $y(1) = 0$ then tells us $B = -2/3$, and therefore have obtained two different solutions

$$y = \left(\frac{2}{3} x - \frac{2}{3} \right)^{3/2}$$

and also

$$y = -\left(\frac{2}{3} x - \frac{2}{3} \right)^{3/2}$$

both satisfying the initial condition. In fact $y = 0$ is a third solution.