Calculus Stuff to Know

Intro. to Differential Equations

January 11, 2016

1 Integration Techniques

Your success in this class, as well as your ability to enjoy the material in this class (yes, enjoy a math class!), depends strongly on your ability to apply many of the integration techniques that you learned in your study of calculus. In this section we present a nonexhaustive list of techniques that it will be important that you know. They are guaranteed to show up in the homework, on quizzes, and the exams – repeatedly. As such, not taking the time to review them now would be a great way to damage your performance in this class. Don't be that person!

The explanation below is very terse, since it is assumed that students are already quite familiar with the material, and just need a little bit of reminding.

1.1 Integration by Parts

Integration by parts tells us that

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

This can be useful in many situations, as shown by the following example.

Example 1. In this example, we will use integration by parts to calculate the integral $\int xe^x dx$. To do so, we take $g(x) = x$ and $f(x) = e^x$, so that $\int f'(x)g(x)dx = \int xe^x dx$. Then by the above expression, we have

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

Example 2. In this example, we will use integration by parts to calculate the integral $\int \ln(x)dx$. To do so, we take $g(x) = \ln(x)$ and $f(x) = x$, so that $\int f'(x)g(x)dx = \int \ln(x)dx$. Then by the above expression, we have

$$\int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - x + C.$$

Exercise 1. Determine the following integrals:

(a) $\int x \sin(x) dx$

(b) $\int x^2 e^{3x} dx$

(c) $\int x \ln(x) dx$

(d) $\int \cos(2x) e^{4x}$

1.2 Partial Fraction Decomposition

Recall that a rational function is a function of the form

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n},$$

ie. a polynomial divided by a polynomial. The method of integrating rational functions is reduce the rational function to a *standard form*, where it becomes a linear combination of functions that we know how to integrate. This method of reducing to a standard form is called Partial Fraction Decomposition (PFD). The steps to decompose something to partial fraction decomposition form are always the same:

Step 1: If the degree of the numerator is greater than or equal to the degree of the denominator, then use polynomial long division to write $f(x)$ as a polynomial plus a rational function whose numerator has degree strictly less than the denominator.

Step 2: Factor the denominator of the remaining rational function as a product of *irreducible polynomials*

Step 3: Write the remaining rational function as a linear combination of elementary rational functions with undetermined coefficients – the elementary rational functions that you choose depend on the factorization of the denominator that you found earlier.

Step 4: Solve for the undetermined coefficients

The steps where students most often screw up are Step 3 (they don't choose the right standard form) and Step 4 (they screw up the algebra). Step 3 is a matter of memorization – you need to remember the right thing to do. Getting Step 4 right can be tricky. One way students can help themselves with this part is to always be thinking "is there an easier way to do this"? In other words, if you can do something clever and concise, rather than a long an laborious calculation, then it is less likely for you to make a computational error halfway through.

1.2.1 Standard Forms

The standard form for a rational function is as a linear combination of *elementary* rational functions – rational functions which are considered as reduced as possible. Examples of elementary rational functions are:

$$\frac{1}{x}, \frac{1}{x-1}, \frac{1}{(x-2)^2}, \frac{2x-3}{x^2+x+1}.$$

The actual definition of an elementary rational function is the following:

Definition 1. A *elementary rational function* is a rational function that is of the form

$$\frac{a}{(x+c)^n}$$

for some real constants a, c and some nonzero integer n, or of the form

$$\frac{ax+d}{(x^2+bx+c)^n}$$

for some real constants a, b, c, d and some integer n, where x^2+bx+c is an *irreducible* polynomial (e.g. has no real roots).

Note that according to this definition, the rational function $\frac{2x+3}{x^2+1}$ is an elementary rational function, because the polynomial x^2+1 is irreducible. The rational function $\frac{2x+3}{x^2-1}$ is NOT an elementary rational function, because the polynomial x^2-1 may be reduced: $x^2-1 = (x-1)(x+1)$.

Let's now suppose that we've done Step 1 and Step 2, and we need to write down the standard form of the remaining rational function with factored denominator. The rules seem a bit arbitrary, and they occur for algebraic reasons that we won't go into. In the end, choosing the right standard form is really a matter of memorization. Rather than give a general characterization, we start off with a couple of examples.

Example 3. The standard form for $\frac{3x-2}{(x-3)(x-4)}$ is some linear combination of elementary rational functions. Which ones? To decide this, we look at the irreducible polynomials occuring in the denominator, namely $(x-3)$ and $(x-4)$. From these, we know can automatically say that the standard form will be a linear combination of the elementary rational functions $\frac{1}{(x-3)}$ and $\frac{1}{(x-4)}$. In other words:

$$\frac{3x-2}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4},$$

for some constants A and B. Some algebra then allows us to determine the values of A and B explicitly (which is Step 4). Let's do this, for purposes of our demonstration. Multiplying both sides of the above equation by the denominator of the left hand side, we find

$$3x-2 = A(x-4) + B(x-3).$$

When $x=4$, this says $3 \cdot 4 - 2 = A(4-4) + B(4-3)$, eg. $10 = B$. Now we know the value of B! Next, when $x=3$ this says $3 \cdot 3 - 2 = A(3-4) + B(3-3)$, eg. $7 = -A$. Thus $A = -7$. To conclude, we've found

$$\frac{3x-2}{(x-3)(x-4)} = \frac{-7}{x-3} + \frac{10}{x-4},$$

and in this way we've expressed our original rational function in standard form.

Example 4. The standard form for $\frac{3x-2}{(x-3)^2}$ is some linear combination of elementary rational functions. Which ones? To decide this, we look at the irreducible polynomials occuring in the denominator, namely $(x-3)$ which occurs twice. From these, we know can automatically say that the standard form will be a linear combination of the elementary rational functions $\frac{1}{(x-3)}$ and $\frac{1}{(x-3)^2}$. In other words:

$$\frac{3x-2}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2},$$

for some constants A and B. Some algebra then allows us to determine the values of A and B explicitly (which is Step 4). Let's do this, for purposes of our demonstration. Multiplying both sides of the above equation by the denominator of the left hand side, we find

$$3x - 2 = A(x-3) + B.$$

When $x = 3$, this says $3 \cdot 3 - 2 = A(3-3) + B$, eg. $7 = B$. Now we know the value of B! To determine the value of A, we can compare the coefficient of x on both sides. On the left hand side, the coefficient is 3, and on the right hand side the coefficient is A. Therefore $A = 3$. To conclude, we've found

$$\frac{3x-2}{(x-3)^2} = \frac{3}{x-3} + \frac{7}{(x-3)^2},$$

and in this way we've expressed our original rational function in standard form.

Here are some more examples of the standard forms of rational functions (we leave solving for the corresponding constants as an exercise for the reader):

Example 5.

$$\frac{2x-3}{(x-1)^2(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}.$$

Example 6.

$$\frac{x+17}{(x+4)^3} = \frac{A}{(x+4)} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3}.$$

Example 7.

$$\frac{6x-47}{(x^2+x+1)(x-3)} = \frac{A}{(x-3)} + \frac{Bx+C}{x^2+x+1}.$$

Example 8.

$$\frac{2x-2}{(x^2+x+1)^2(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}.$$

To solve for the coefficients in the above again is simply a matter of algebra. The wisest students will find the right combination of considering various nice values of x and comparing coefficients to make the job of calculating the coefficients as easy and painless as possible. Remember, the less work you have to do, the harder it is to make a computational error.

1.2.2 Using PFD to Integrate

Partial fraction decomposition is useful to us in calculus, because it allows us to integrate rational functions. The idea is to take whatever rational function you are interested in, use PFD to put it in standard form, and then integrate the corresponding elementary rational functions individually. This latter task may involve some u-substitution or possibly trig substitution.

Example 9. Calculate the integral

$$\int \frac{x^2 + 2x + 3}{x^2 - 1} dx.$$

To do this, we first need to put the rational function into standard form. Note that the numerator has the same degree as the denominator, so we need to start out with long division. Doing so, we find:

$$\frac{x^2 + 2x + 3}{x^2 - 1} = 1 + \frac{2x + 4}{x^2 - 1}.$$

Next, the denominator factors as $(x - 1)(x + 1)$, and therefore we know

$$\frac{2x + 4}{x^2 - 1} = \frac{2x + 4}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1}.$$

Multiplying by $(x - 1)(x + 1)$, this says

$$2x + 4 = A(x + 1) + B(x - 1).$$

When $x = -1$, this says $2 = -2B$, and therefore $B = -1$. When $x = 1$, this says $6 = 2A$ and therefore $A = 3$. We conclude

$$\frac{2x + 4}{x^2 - 1} = \frac{3}{x - 1} + \frac{-1}{x + 1}.$$

In this way, we see

$$\int \frac{x^2 + 2x + 3}{x^2 - 1} dx = \int 1 + \frac{3}{x - 1} + \frac{-1}{x + 1} dx = x + 3 \ln |x - 1| - \ln |x + 1| + C.$$

Exercise 2. Determine the following integrals:

(a) $\int \frac{x}{(x-1)^2} dx$

(b) $\int \frac{x}{x^2+x+1} dx$

(c) $\int \frac{1}{x(x^2-x+1)} dx$

(d) $\int \frac{2x^2+3x-4}{(x-1)(x+3)} dx$

2 Miscellaneous Integrals

Here's a random collection of integrals that you should just know.

$$\int \tan(x)dx = -\ln|\cos(x)| + C.$$

$$\int \cot(x)dx = \ln|\sin(x)| + C.$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

$$\int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C.$$

3 Trigonometric Identities

Integration, differentiation, etc. is often facilitated by the timely use of trigonometric identities. There are a few identites that have such phenomenal cosmic power that you should remember them for all time. These are listed here, and they *will* be useful in this class, as well as many other occasions.

The most important trig. identity is the following:

$$\sin^2(x) + \cos^2(x) = 1.$$

The other two identities that you should remember are the angle addition formulai

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Note that most of the other trig identities that you know and love are simply a consequence of these three. For example, if we divide the first identity on both sides by $\cos^2(x)$, then we get the trig. identity

$$\tan^2(x) + 1 = \sec^2(x).$$

Alternatively if we divide by $\sin^2(x)$, then we get the trig. identity

$$1 + \cot^2(x) = \csc^2(x).$$

Furthermore, if we take $\alpha = \beta$ in the angle addition formula for cosine, then we see

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha).$$

If we then use the fact that $\sin^2(\alpha) = 1 - \cos^2(\alpha)$, then we find

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1,$$

and solving for $\cos^2(\alpha)$, we obtain

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha),$$

which is the usual double-angle formula!