Math 308 Final Exam Sample Questions

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1 Introduction

The final exam is almost upon us. Jeepers! In preparing for the final exam, there are several things to keep in mind

- the final is cumulative
- definitions will make up a significant portion of the final

Aside from knowing the exact statements of definitions, you should also know the major results/theorems/techniques. Even further, you should be able to apply the major theorems to solve problems. This document is intended to give you a sampling of problems similar to those you should expect to run into on the final exam, aside from the questions asking that you state definitions. However, this sampling of problems should not be considered to be "exhaustive", and the actual final exam may contain some completely different questions or question types. Therefore it is also encouraged that you seek additional questions elsewhere, both in the book and online. In particular, a search on the internet may lead you to several examples of past Math 308 exams. Here, the problems are broken down into three types: calculation based questions, true/false questions, conceptual questions.

2 Calculation-Type Questions

Question 1. Suppose that A is the 3×4 matrix

$$A = \begin{bmatrix} -1 & 4 & -3 & 4 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and that

$$\vec{b} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

(a) Find a basis for $\mathcal{N}(A)$

(b) Check that

$$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

is a solution to the equation $A\vec{x} = \vec{b}$

(c) Using (a) and (b), determine *all* solutions to the equation $A\vec{x} = \vec{b}$

Question 2. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{bmatrix}$$

(a) Determine a basis for $\mathcal{R}(A)$

(b) Use Gram-Schmidt to turn the basis for $\mathcal{R}(A)$ found in (a) into an orthonormal basis

Question 3. Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}.$$

Question 4. Calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & -1 & 5 & 0 \\ 2 & -2 & 11 & 2 \\ 0 & 2 & 8 & 1 \end{bmatrix}.$$

Question 5. Suppose that we are given a table of data

x	1	2	3	4
y	-2	3	7	10

Find constants m, b so that the equation

$$y = mx + b$$

most closely approximates the data in the table. (hint: use least squares)

Question 6. Suppose that f is a linear function from $\mathbb{R}^3$ to $\mathbb{R}^4$ satisfying

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

(a) Find a 4×3 matrix A satisfying $f(\vec{v}) = A\vec{v}$.

(b) Find a basis for the null space of A

(c) Determine the rank and nullity of A

Question 7. Let A be the matrix defined by

$$A = \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix}$$

1. Find all the eigenvalues of A

2. For each eigenvalue, compute a basis of the corresponding eigenspace

3. Determine the algebraic and geometric multiplicity of each eigenvalue of A

4. Let

$$\vec{v} = \begin{bmatrix} 1 \\ 11 \\ 5 \end{bmatrix}.$$

Using (b), determine the value of $A^{10}\vec{v}$. (hint: expand $\vec{v}$ in terms of the various eigenvectors)

Question 8. Consider the matrix A defined by

$$A = \begin{bmatrix} -7 & 4 & -3 \\ 8 & -3 & 3 \\ 32 & -16 & 13 \end{bmatrix}$$

(a) Calculate the characteristic polynomial of A

(b) Use (a) to show that 1 is an eigenvalue, and calculate its algebraic multiplicity

(c) Find a basis for E_1, and determine the geometric multiplicity of 1

Question 9. Write down an example of a matrix which is not diagonal, but is similar to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Question 10. Find a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ which takes the standard basis to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Question 11. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and the basis

$$\mathcal{A} = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$

(b) Write the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}}$$

in terms of the basis $\mathcal{A}$.

Question 12. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $f(\vec{x}) = A\vec{x}$ for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Write a matrix representation for f in terms of the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$ and

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

for $\mathbb{R}^2$.

3 True-False Type Questions

For each of the following questions, determine whether the statement is TRUE or FALSE.

TF 1 Let A, B, C be $n \times n$ matrices. If A is similar to C and B is similar to C, then A is similar to B.

TF 2 If A is an $n \times n$ matrix, and $A^m = I$ for some integer n, then $A^{-1} = I + A + A^2 + \cdots + A^{m-1}$

TF 3 Suppose that A and B are two $m \times n$ matrices, and that $\vec{b}$ is a vector in $\mathbb{R}^m$. Then the systems of equations $A\vec{x} = \vec{b}$ and $B\vec{x} = \vec{b}$ have the same set of solutions if and only if A and B have the same RREF.

TF 4 If A is a matrix and A^2 is the zero matrix, then A is the zero matrix.

TF 5 If A is an $m \times n$ matrix, and B, C are $\ell \times m$ matrices, and $BA = CA$, then $B = C$.

TF 6 Every set of nonzero orthogonal vectors in $\mathbb{R}^n$ is linearly independent.

TF 7 Every nontrivial subspace of $\mathbb{R}^n$ has an orthogonal basis.

TF 8 Every nontrivial subspace of $\mathbb{R}^n$ has an orthonormal basis.

TF 9 A subset of a linearly independent set of vectors is linearly independent.

TF 10 Any set of linearly independent vectors in a subspace V of $\mathbb{R}^n$ can be extended to a basis for V.

TF 11 Any set of vectors in a subspace V of $\mathbb{R}^n$ that spans V also contains a basis for V.

TF 12 Let X and Y be subsets of $\mathbb{R}^n$, with $X \subseteq Y$, and suppose that X is linearly independent. Then Y is linearly indepdentent

TF 13 If V and W are subspaces of $\mathbb{R}^n$, then so too is the intersection $V \cap W$

TF 14 If V and W are subspaces of $\mathbb{R}^n$, then so too is the union $V \cup W$

TF 15 The linear system of equations $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ may be expressed as a linear combination of the column vectors of A

TF 16 If A, B, C are all $n \times n$ matrices, then $A(BC) = (AB)C$

TF 17 If A, B are both $n \times n$ matrices, then $AB = BA$

TF 18 If A, B are both $n \times n$ matrices, then $(AB)^T = B^T A^T$

TF 19 If A is a 2×3 matrix and B is a 3×2 matrix, then it is possible for AB to be the 2×2 identity matrix

TF 20 If A is a 2×3 matrix and B is a 3×2 matrix, then it is possible for BA to be the 3×3 identity matrix

TF 21 If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$

TF 22 If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$

TF 23 If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$

TF 24 If A and B are $n \times n$ matrices, then $\det(A^T) = -\det(A)$

TF 25 If A is an $n \times n$ matrix and $\vec{v}$ is an eigenvector of A with eigenvalue 7, then $3\vec{v}$ is an eigenvector of A with eigenvalue 21

TF 26 The only matrix that is similar to the identity matrix is the identity matrix.

TF 27 If A is a square matrix, then A is similar to $-A$.

TF 28 If λ is an eigenvalue of A, then λ^m is an eigenvalue of A^m.

TF 29 If D is a diagonal matrix, and λ is an eigenvalue of D, then the algebraic and geometric multiplicities of λ are the same.

TF 30 If A and C are similar matrices, then A and C have the same eigenvalues, with the same multiplicities (both algebraic and geometric).

TF 31 Zero is never an eigenvalue of a matrix.

TF 32 If A is a matrix with eigenvalue λ, then $\lambda^2 + 2\lambda - 17$ is an eigenvalue of the matrix $A^2 + 2A - 17I$.

TF 33 The span of any three vectors in $\mathbb{R}^4$ is a three dimensional subspace of $\mathbb{R}^4$.

TF 34 If the nullity of A is zero, then the linear homogeneous system of equations $A\vec{x} = \vec{0}$ has infinitely many solutions.

TF 35 If A is an $n \times n$ matrix with two identical rows, then $\det(A) = 0$

TF 36 If A is an $n \times n$ matrix with two identical columns, then $\det(A) = 0$

TF 37 If A is a 4×3 matrix, with $\text{null}(A) = 1$, then $\text{rank}(A) = 2$.

TF 38 If A is a square matrix, and $B = A^3 - 27A^2 + 16A - I$, then $AB = BA$

TF 39 If A is similar to a diagonalizable matrix, then A is diagonalizable

4 More Conceptual Questions

Question 13. Suppose that W is the subset of $\mathbb{R}^2$ given by

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \text{ real number}, \ x_1 x_2 = 0 \right\}.$$

Prove that W is not a subspace of $\mathbb{R}^2$. What closure property fails?

Question 14. Suppose that W is the subset of $\mathbb{R}^2$ given by

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \text{ integers} \right\}.$$

Prove that W is not a subspace of $\mathbb{R}^2$. What closure property fails?

Question 15. Let $a_0, \ldots, a_n$ be constants, not all zero. Show that the set

$$W = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \text{ real numbers}, \ a_1 x_1 + \cdots + a_n x_n = 0 \right\}.$$

is a subspace of $\mathbb{R}^n$, and show that it's dimension is exactly $n-1$. (hint: this can be done in a very short one or two sentences, by using one of our major theorems)

Question 16. Suppose that V is a subspace of $\mathbb{R}^n$, and that $\{\vec{v}_1, \ldots, \vec{v}_d\}$ is an orthonormal basis for V. Show that the only vector $\vec{v}$ in V satisfying $\vec{v}_j \cdot \vec{v} = 0$ for all $1 \leq j \leq d$ is the zero vector.

Question 17. Show that if an $n \times n$ invertible matrix A is diagonalizable, then so too is A^{-1}.

Question 18. A square matrix P is called *idempotent* if $P^2 = P$. Show that the only invertible idempotent matrix is the identity matrix. (hint: do a little matrix algebra on the equation $P^2 - P = 0$)

Question 19. Find three 2×2 matrices A satisfying the equation $A^2 = I$, none of which are similar to each other

Question 20. Write down an example of a matrix which is not diagonalizable.

Question 21. Write down an example of a matrix (other than $\pm I$) that is orthogonal.

Question 22. Show that if A is an $n \times n$ orthogonal matrix, then the column vectors of A are orthogonal.

Question 23. Suppose that A is a symmetric matrix, and that $\vec{v}$ and $\vec{w}$ are eigenvectors of A with eigenvalues λ and ω which are not the same. Show that $\vec{v} \perp \vec{w}$. (hint: think about $(A\vec{v}) \cdot \vec{w}$ and $\vec{v} \cdot (A\vec{w})$)