Math 308 Final Exam Sample Questions

W.R. Casper

December 8, 2013

1 Introduction

The final exam is almost upon us. Jeepers! In preparing for the final exam, there are several things to keep in mind

- the final is cumulative
- definitions will make up a significant portion of the final

Aside from knowing the exact statements of definitions, you should also know the major results/theorems/techniques. Even further, you should be able to apply the major theorems to solve problems. This document is intended to give you a sampling of problems similar to those you should expect to run into on the final exam, aside from the questions asking that you state definitions. However, this sampling of problems should not be considered to be "exhaustive", and the actual final exam may contain some completely different questions or question types. Therefore it is also encouraged that you seek additional questions elsewhere, both in the book and online. In particular, a search on the internet may lead you to several examples of past Math 308 exams. Here, the problems are broken down into three types: calculation based questions, true/false questions, conceptual questions.

2 Calculation-Type Questions

Question 1. Suppose that A is the 3×4 matrix

$$A = \begin{bmatrix} -1 & 4 & -3 & 4 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and that

$$\vec{b} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

(a) Find a basis for $\mathcal{N}(A)$

(b) Check that

$$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

is a solution to the equation $A\vec{x} = \vec{b}$

(c) Using (a) and (b), determine *all* solutions to the equation $A\vec{x} = \vec{b}$

Solution 1.

(a) We first determine the RREF of A, which turns out to be

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This tells us we have one free variable, namely x_3, and the null space $\mathcal{N}(A)$ is

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -5x_3 \\ -(1/2)x_3 \\ x_3 \\ 0 \end{bmatrix} : x_3 \text{ a real number} \right\} = \text{span}\left(\left\{ \begin{bmatrix} -5 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} \right\} \right).$$

Thus a basis for $\mathcal{N}(A)$ consists of the single vector $\begin{bmatrix} -5 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$.

(b) Matrix multiplication of A with $\vec{y}$ shows that $A\vec{y} = \vec{b}$

(c) Using (a) and (b), any solution to $A\vec{x} = \vec{b}$ must be of the form

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$$

for some constant c.

Question 2. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{bmatrix}$$

(a) Determine a basis for $\mathcal{R}(A)$ consisting of column vectors of A

(b) Use Gram-Schmidt to turn the basis for $\mathcal{R}(A)$ found in (a) into an orthonormal basis

Solution 2.

(a) Let $\vec{A}_1, \vec{A}_2, \vec{A}_3, \vec{A}_4$ be the column vectors of A. We know that

$$\mathcal{R}(A) = \operatorname{span}\{\vec{A}_1, \vec{A}_2, \vec{A}_3, \vec{A}_4\},$$

and therefore by removing some of the column vectors, we should get a basis for $\mathcal{R}(A)$. We first calculate the RREF of A, which turns out to be

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This shows us that the null space of A is

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} 2x_3 \\ -3x_3 \\ x_3 \\ 0 \end{bmatrix} : x_3 \text{ a real number} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

In particular, this means that the nullity of A is 1, so that the rank of A is $4 - 1 = 3$ by the rank-nullity theorem. Therefore a basis for $\mathcal{R}(A)$

will have precisely three vectors. Thus to get a basis for $\mathcal{R}(A)$, we need to remove exactly 1 column vector from the set $\{\vec{A}_1, \vec{A}_2, \vec{A}_3, \vec{A}_4\}$. Since $[2, -3, 1, 0]^T$ is in the kernel of A, we know that $2\vec{A}_1 - 3\vec{A}_2 + \vec{A}_3 = \vec{0}$. This means that $\vec{A}_3$ is in the span of $\{\vec{A}_1, \vec{A}_2\}$. Hence we can remove it to get a basis $\{\vec{A}_1, \vec{A}_2, \vec{A}_4\}$ for $\mathcal{R}(A)$.

(b) We start with $\vec{A}_1$, which we normalize to get the vector

$$\vec{u}_1 = (1/\|\vec{A}_1\|)\vec{A}_1 = \begin{bmatrix} 0.40825 \\ -0.40825 \\ 0.81650 \end{bmatrix}.$$

We next form the vector

$$\vec{v}_2 = \vec{A}_2 - (\vec{A}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$$

and normalize it to get

$$\vec{u}_2 = (1/\|\vec{v}_2\|)\vec{v}_2 = \begin{bmatrix} -0.26726 \\ 0.80178 \\ 0.53452 \end{bmatrix}$$

Lastly we form the vector

$$\vec{v}_3 = \vec{A}_4 - (\vec{A}_4 \cdot \vec{u}_1)\vec{u}_1 - (\vec{A}_4 \cdot \vec{u}_2)\vec{u}_2 = \begin{bmatrix} -0.57143 \\ -0.28571 \\ 0.14286 \end{bmatrix}$$

and normalize to get

$$\vec{u}_3 = (1/\|\vec{v}_3\|)\vec{v}_3 = \begin{bmatrix} -0.87287 \\ -0.43644 \\ 0.21822 \end{bmatrix}$$

Then $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis for $\mathcal{R}(A)$.

Question 3. Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}.$$

Solution 3. We determine the inverse by row reducing the augmented matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right].$$

Doing so, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right].$$

Therefore, the inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}.$$

Question 4. Calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & -1 & 5 & 0 \\ 2 & -2 & 11 & 2 \\ 0 & 2 & 8 & 1 \end{bmatrix}.$$

Solution 4. We can calculate the determinant of A straight-forwardly, using the definition of the determinant. However, that sounds like a pain. Instead, we'll first simplify the problem by using a sequence of elementary row operations. The strategy is simple: the determinant of an upper triangular matrix is exactly the product of the elements on the main diagonal; if we use elementary row operations to transform A to an upper triangular matrix, and keep track of the operations we do, then we can easily calculate the determinant!

We proceed as follows:

$$R_2 - R_1 \longrightarrow \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 2 & -2 & 11 & 2 \\ 0 & 2 & 8 & 1 \end{bmatrix}$$

$$R_3 - 2R_1 \longrightarrow \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 7 & 0 \\ 0 & 2 & 8 & 1 \end{bmatrix}$$

$$R_1 + 2R_2 \longrightarrow \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 7 & 0 \\ 0 & 2 & 8 & 1 \end{bmatrix}$$

$$R_3 - 2R_2 \longrightarrow \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 8 & 1 \end{bmatrix}$$

$$R_4 - 2R_2 \longrightarrow \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

$$R_4 - 2R_3 \longrightarrow \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now recall how elementary row operations interact with determinants. Adding a multiple of row to another doesn't change the determinant. Since all of the operations were adding a multiple of some row to another, we see that in fact

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -1.$$

Question 5. Suppose that we are given a table of data

x	1	2	3	4
y	-2	3	7	10

Find constants m, b so that the equation

$$y = mx + b$$

most closely approximates the data in the table. (hint: use least squares)

Solution 5. The equation $y = mx + b$ along with the data table gives us the four linear equations

$$\begin{aligned} -2 &= m + b \\ 3 &= 2m + b \\ 7 &= 3m + b \\ 10 &= 4m + b \end{aligned}$$

We convert this system to matrix form:

$$A \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 7 \\ 10 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}.$$

This system is inconsistent. A least-squares solution to this is given by solving the consistent system

$$A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \begin{bmatrix} -2 \\ 3 \\ 7 \\ 10 \end{bmatrix},$$

or rather

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 65 \\ 18 \end{bmatrix}.$$

The matrix $\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$ is invertible, and its inverse is $\begin{bmatrix} 1/5 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$. Therefore a least squares solution is

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1/5 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 65 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ -10/2 \end{bmatrix}$$

Thus the closest fit to the data is $y = 4x - 10/2$.

Question 6. Suppose that f is a linear function from $\mathbb{R}^3$ to $\mathbb{R}^4$ satisfying

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

(a) Find a 4×3 matrix A satisfying $f(\vec{v}) = A\vec{v}$.

(b) Find a basis for the null space of A

(c) Determine the rank and nullity of A

Solution 6.

(a) In order to find A, we need to determine where f sends the standard basis vectors $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$. First note that

$$\vec{e}_1 = (1/2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - (1/2) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (1/2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and therefore since f is linear

$$f(\vec{e}_1) = (1/2)f\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - (1/2)f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) + (1/2)f\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$
$$= (1/2)\begin{bmatrix}1\\2\\3\\4\end{bmatrix} - (1/2)\begin{bmatrix}2\\4\\6\\8\end{bmatrix} + (1/2)\begin{bmatrix}4\\3\\2\\1\end{bmatrix} = \begin{bmatrix}3/2\\1/2\\-1/2\\-3/2\end{bmatrix}$$

Similarly,

$$\vec{e}_2 = (1/2)\begin{bmatrix}1\\1\\0\end{bmatrix} + (1/2)\begin{bmatrix}0\\1\\1\end{bmatrix} - (1/2)\begin{bmatrix}1\\0\\1\end{bmatrix}$$

and since f is linear

$$f(\vec{e}_2) = (1/2)f\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + (1/2)f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) - (1/2)f\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$
$$= (1/2)\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + (1/2)\begin{bmatrix}2\\4\\6\\8\end{bmatrix} - (1/2)\begin{bmatrix}4\\3\\2\\1\end{bmatrix} = \begin{bmatrix}-1/2\\3/2\\7/2\\11/2\end{bmatrix}$$

and also

$$\vec{e}_3 = -(1/2)\begin{bmatrix}1\\1\\0\end{bmatrix} + (1/2)\begin{bmatrix}0\\1\\1\end{bmatrix} + (1/2)\begin{bmatrix}1\\0\\1\end{bmatrix}$$

and therefore since f is linear

$$f(\vec{e}_3) = -(1/2)f\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + (1/2)f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) + (1/2)f\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$
$$= -(1/2)\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + (1/2)\begin{bmatrix}2\\4\\6\\8\end{bmatrix} + (1/2)\begin{bmatrix}4\\3\\2\\1\end{bmatrix} = \begin{bmatrix}5/2\\5/2\\5/2\\5/2\end{bmatrix}$$

If $f(\vec{v}) = A\vec{v}$, then in particular $f(\vec{e}_j) = A\vec{e}_j$, so $f(\vec{e}_j)$ must be the j'th column of A. This tells us

$$A = \begin{bmatrix} 3/2 & -1/2 & 5/2 \\ 1/2 & 3/2 & 5/2 \\ -1/2 & 7/2 & 5/2 \\ -3/2 & 11/2 & 5/2 \end{bmatrix}$$

(b) We calculate the null space of A in the usual way, finding

$$\mathcal{N}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\},$$

so that a basis for $\mathcal{N}(A)$ is the set with one vector

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(c) Part (b) shows us that the nullity of A is 1. Therefore by the rank nullity theorem, the rank of A must be $3 - 1 = 2$.

Question 7. Let A be the matrix defined by

$$A = \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix}$$

1. Find all the eigenvalues of A
2. For each eigenvalue, compute a basis of the corresponding eigenspace
3. Determine the algebraic and geometric multiplicity of each eigenvalue of A
4. Let
$$\vec{v} = \begin{bmatrix} 1 \\ 11 \\ 5 \end{bmatrix}.$$
Using (b), determine the value of $A^{10}\vec{v}$. (hint: expand $\vec{v}$ in terms of the various eigenvectors)

Solution 7.

(a) We calculate $\det(A - tI) = -(t-3)^2(t+1)$. Therefore the eigenvalues of A are -1 and 3

(b) We first calculate the RREF of $A - (-1)I = A + I$ to be

$$\begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$E_{-1} = \mathcal{N}(A+I) = \operatorname{span}\left\{ \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix} \right\}$$

so that E_{-1} is one dimensional with basis

$$\left\{ \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix} \right\}$$

We next calculate the RREF of $A - 3I$ to be

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$E_3 = \mathcal{N}(A - 3I) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

so that E_2 is two dimensional with basis

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) Parts (a) and (b) show that

- eigenvalue -1 has algebraic multiplicity 1 and geometric multiplicity 1

- eigenvalue 3 has algebraic multiplicity 2 and geometric multiplicity 2

(d) We find that

$$\vec{v} = 15 \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix} - 9 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 10 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore

$$\begin{aligned} A^{10}\vec{v} &= 15A^{10} \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix} - 9A^{10} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 10A^{10} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= 15(-1)^{10} \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix} - 9(3)^{10} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 10(3)^{10} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -531431 \\ -531421 \\ -590475 \end{bmatrix} \end{aligned}$$

Question 8. Consider the matrix A defined by

$$A = \begin{bmatrix} -7 & 4 & -3 \\ 8 & -3 & 3 \\ 32 & -16 & 13 \end{bmatrix}$$

(a) Calculate the characteristic polynomial of A

(b) Use (a) to show that 1 is an eigenvalue, and calculate its algebraic multiplicity

(c) Find a basis for E_1, and determine the geometric multiplicity of 1

Solution 8.

(a) We calculate $\det(A - tI) = -(t-1)^3$

(b) Since the eigenvalues of A are exactly the roots of the characteristic polynomial, this shows that 1 is an eigenvalue (and in fact the only eigenvalue). Its algebraic multiplicity is 3.

(c) We calculate the RREF of $A - \lambda I$ with $\lambda = 1$ to be

$$A = \begin{bmatrix} 1 & -1/2 & 3/8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the null space of $A - I$ is

$$E_1 = \mathcal{N}(A - I) = \text{span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

It's easy to check that the set

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent. Since it spans E_1, this is a basis for E_1. This also shows that the algebraic multiplicity of λ is 2.

Question 9. Write down an example of a matrix A which is not diagonal, but is similar to the matrix $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Solution 9. Easy! Just pick a (nondiagonal) invertible matrix B and set $A = BCB^{-1}$. Then A and C will be similar (clearly) and A will not be diagonal. Now let's do this explicitly. Take $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then

$$A = BCB^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}.$$

is similar to C.

3 True-False Type Questions

For each of the following questions, determine whether the statement is TRUE or FALSE.

TF 1 Let A, B, C be $n \times n$ matrices. If A is similar to C and B is similar to C, then A is similar to B. (TRUE)

TF 2 If A is an $n \times n$ matrix, and $A^m = I$ for some integer n, then $A^{-1} = I + A + A^2 + \cdots + A^{m-1}$ (FALSE)

TF 3 Suppose that A and B are two $m \times n$ matrices, and that $\vec{b}$ is a vector in $\mathbb{R}^m$. Then the systems of equations $A\vec{x} = \vec{b}$ and $B\vec{x} = \vec{b}$ have the same set of solutions if and only if A and B have the same RREF. (FALSE)

TF 4 If A is a matrix and A^2 is the zero matrix, then A is the zero matrix. (FALSE)

TF 5 If A is an $m \times n$ matrix, and B, C are $\ell \times m$ matrices, and $BA = CA$, then $B = C$. (FALSE)

TF 6 Every set of nonzero orthogonal vectors in $\mathbb{R}^n$ is linearly independent. (TRUE)

TF 7 Every nontrivial subspace of $\mathbb{R}^n$ has an orthogonal basis. (TRUE)

TF 8 Every nontrivial subspace of $\mathbb{R}^n$ has an orthonormal basis. (TRUE)

TF 9 A subset of a linearly independent set of vectors is linearly independent. (TRUE)

TF 10 Any set of linearly independent vectors in a subspace V of $\mathbb{R}^n$ can be extended to a basis for V. (TRUE)

TF 11 Any set of vectors in a subspace V of $\mathbb{R}^n$ that spans V also contains a basis for V. (TRUE)

TF 12 Let X and Y be subsets of $\mathbb{R}^n$, with $X \subseteq Y$, and suppose that X is linearly independent. Then Y is linearly independent (FALSE)

TF 13 If V and W are subspaces of $\mathbb{R}^n$, then so too is the intersection $V \cap W$ (TRUE)

TF 14 If V and W are subspaces of $\mathbb{R}^n$, then so too is the union $V \cup W$ (FALSE)

TF 15 The linear system of equations $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ may be expressed as a linear combination of the column vectors of A (TRUE)

TF 16 If A, B, C are all $n \times n$ matrices, then $A(BC) = (AB)C$ (TRUE)

TF 17 If A, B are both $n \times n$ matrices, then $AB = BA$ (FALSE)

TF 18 If A, B are both $n \times n$ matrices, then $(AB)^T = B^T A^T$ (TRUE)

TF 19 If A is a 2×3 matrix and B is a 3×2 matrix, then it is possible for AB to be the 2×2 identity matrix (TRUE)

TF 20 If A is a 2×3 matrix and B is a 3×2 matrix, then it is possible for BA to be the 3×3 identity matrix (FALSE)

TF 21 If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$ (TRUE)

TF 22 If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$ (FALSE)

TF 23 If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$ (TRUE)

TF 24 If A and B are $n \times n$ matrices, then $\det(A^T) = -\det(A)$ (FALSE)

TF 25 If A is an $n \times n$ matrix and $\vec{v}$ is an eigenvector of A with eigenvalue 7, then $3\vec{v}$ is an eigenvector of A with eigenvalue 21 (FALSE)

TF 26 The only matrix that is similar to the identity matrix is the identity matrix. (TRUE)

TF 27 If A is a square matrix, then A is similar to $-A$. (FALSE)

TF 28 If λ is an eigenvalue of A, then λ^m is an eigenvalue of A^m. (TRUE)

TF 29 If D is a diagonal matrix, and λ is an eigenvalue of D, then the algebraic and geometric multiplicities of λ are the same. (TRUE)

TF 30 If A and C are similar matrices, then A and C have the same eigenvalues, with the same multiplicities (both algebraic and geometric). (TRUE)

TF 31 Zero is never an eigenvalue of a matrix. (FALSE)

TF 32 If A is a matrix with eigenvalue λ, then $\lambda^2 + 2\lambda - 17$ is an eigenvalue of the matrix $A^2 + 2A - 17I$. (TRUE)

TF 33 The span of any three vectors in $\mathbb{R}^4$ is a three dimensional subspace of $\mathbb{R}^4$. (FALSE)

TF 34 If the nullity of A is zero, then the linear homogeneous system of equations $A\vec{x} = \vec{0}$ has infinitely many solutions. (FALSE)

TF 35 If A is an $n \times n$ matrix with two identical rows, then $\det(A) = 0$ (TRUE)

TF 36 If A is an $n \times n$ matrix with two identical columns, then $\det(A) = 0$ (TRUE)

TF 37 If A is a 4×3 matrix, with $\text{null}(A) = 1$, then $\text{rank}(A) = 2$. (TRUE)

TF 38 If A is a square matrix, and $B = A^3 - 27A^2 + 16A - I$, then $AB = BA$ (TRUE)

TF 39 If A is similar to a diagonalizable matrix, then A is diagonalizable (TRUE)

4 More Conceptual Questions

Question 10. Suppose that W is the subset of $\mathbb{R}^2$ given by

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \text{ real number, } \; x_1 x_2 = 0 \right\}.$$

Prove that W is not a subspace of $\mathbb{R}^2$. What closure property fails?

Solution 10. Not closed under addition: $\binom{1}{0}$ and $\binom{0}{1}$ are both vectors in W, but their sum $\binom{1}{1}$ is not in W, since $1 \cdot 1 \neq 0$.

Question 11. Suppose that W is the subset of $\mathbb{R}^2$ given by

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \text{ integers} \right\}.$$

Prove that W is not a subspace of $\mathbb{R}^2$. What closure property fails?

Solution 11. Not closed under scalar multiplication: $\binom{1}{0}$ is in W, but $(3/2)\binom{1}{0} = \binom{3/2}{0}$ is not in W

Question 12. Let $a_0, \ldots, a_n$ be constants, not all zero. Show that the set

$$W = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \text{ real numbers, } a_1 x_1 + \cdots + a_n x_n = 0 \right\}.$$

is a subspace of $\mathbb{R}^n$, and show that it's dimension is exactly $n-1$. (hint: this can be done in a very short one or two sentences, by using one of our major theorems)

Solution 12. Notice that $W = \mathcal{N}(A)$ for A the $1\times n$ matrix $A = [a_1\ a_2\ \ldots\ a_n]$, and therefore is a subspace. Moreover, $\mathcal{R}(A) \subseteq \mathbb{R}^1$, and is not the trivial subspace since at least one of the a_i's is nonzero. Therefore $\mathcal{R}(A) = \mathbb{R}^1$, and in particular is 1-dimensional. The rank-nullity theorem then implies that $\mathcal{N}(A)$ is $n-1$ dimensional.

Question 13. Suppose that V is a subspace of $\mathbb{R}^n$, and that $\{\vec{v}_1, \ldots, \vec{v}_d\}$ is an orthonormal basis for V. Show that the only vector $\vec{v}$ in V satisfying $\vec{v}_j \cdot \vec{v} = 0$ for all $1 \le j \le d$ is the zero vector.

Solution 13. Suppose that $\vec{v} \in V$ satisfies $\vec{v}_j \cdot \vec{v} = 0$ for all $1 \le j \le d$. Since $\{\vec{v}_1, \ldots, \vec{v}_d\}$ is a basis for V and $\vec{v} \in V$, there exist constants $c_1, c_2, \ldots, c_d$ such that

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_d \vec{v}_d.$$

Taking the inner product of both sides of this equation with $\vec{v}_j$, and using the linearity of the inner product, we get

$$\vec{v}_j \cdot \vec{v} = c_1 \vec{v}_j \cdot \vec{v}_1 + c_2 \vec{v}_j \cdot \vec{v}_2 + \ldots c_d \vec{v}_j \cdot \vec{v}_d.$$

Since the basis is orthnormal, $\vec{v}_j \cdot \vec{v}_i = 0$ for $i \neq j$ and 1 for $i = j$. Therefore we have

$$\vec{v}_j \cdot \vec{v} = c_j.$$

However, by assumption $\vec{v}_j \cdot \vec{v} = 0$. Therefore $c_j = 0$. Since j was arbitrary, this must hold for all j. Therefore

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_d \vec{v}_d = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_d = \vec{0}.$$

Question 14. Show that if an $n \times n$ invertible matrix A is diagonalizable, then so too is A^{-1}.

Solution 14. If A is diagonalizable, then there exists a diagonal matrix D and invertible matrix B such that $D = BAB^{-1}$. This also shows that D is equal to a product of three invertible matrices, and must therefore be invertible. Taking the inverse of both sides, we find

$$D^{-1} = (BAB^{-1})^{-1}.$$

Now recall the property of inverses that says that if S, T are two $n \times n$ invertible square matrices, then $(ST)^{-1} = T^{-1}S^{-1}$. Using this, we calculate

$$(BAB^{-1})^{-1} = ((BA)B^{-1})^{-1} = (B^{-1})^{-1}(BA)^{-1} = B(BA)^{-1} = B(A^{-1}B^{-1}) = BA^{-1}B^{-1}.$$

Therefore

$$D^{-1} = BA^{-1}B^{-1}.$$

In particular, this shows that A^{-1} is similar to the diagonal matrix D^{-1}, and is therefore diagonalizable.

Question 15. A square matrix P is called *idempotent* if $P^2 = P$. Show that the only invertible idempotent matrix is the identity matrix. (hint: do a little matrix algebra on the equation $P^2 - P = 0$)

Solution 15. Assume that P is invertible and idempotent. Then $P^2 - P = 0_n$, where the 0_n on the right represents the $n \times n$ zero matrix. Factoring $P^2 - P = P(P - I)$, we see $P(P - I) = 0_n$. Since P is invertible, we can multiply both sides by P^{-1} on the left to find

$$P - I = P^{-1}0_n = 0_n.$$

Hence $P - I = 0_n$, and it follows that $P = I$.

Question 16. Find three 2×2 matrices A satisfying the equation $A^2 = I$, none of which are similar to each other

Solution 16. The three diagonal matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

are all easily seen to satisfy the equation $A^2 = I$. Moreover, none of them have all the same eigenvalues, so none of them are similar.

Question 17. Write down an example of a matrix which is not diagonalizable.

Solution 17. The simplest example is probably

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only eigenvalue of A is 1, with algebraic multiplicity 2. However the geometric multiplicity of eigenvalue 1 for this matrix is 1. Therefore A is defective, and thus cannot diagonalizable.

Question 18. Write down an example of a matrix (other than $\pm I$) that is orthogonal.

Solution 18. For any real value θ, the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(theta) \end{bmatrix}$$

represents the linear transformation that takes a vector in the x, y-plane and rotates it by θ radians counter-clockwise. Since rotation is norm-preserving, this is an orthogonal matrix.

Question 19. Show that if A is an $n \times n$ orthogonal matrix, then the column vectors of A are orthogonal.

Solution 19. Remember: an orthogonal matrix preserves norms:

$$\|A\vec{v}\| = \|\vec{v}\| \text{ for all } \vec{v} \text{ in } \mathbb{R}^n.$$

Now let $\vec{e}_1, \dots, \vec{e}_n$ be the standard basis vectors for $\mathbb{R}^n$. Then for $i \neq j$ we have

$$\|A(\vec{e}_i + \vec{e}_j)\| = \|\vec{e}_i + \vec{e}_j\|,$$

and therefore

$$\|A(\vec{e}_i + \vec{e}_j)\|^2 = \|\vec{e}_i + \vec{e}_j\|^2.$$

Obviously since $i \neq j$ we have $\|\vec{e}_i + \vec{e}_j\|^2 = 2$, and therefore

$$\begin{aligned} 2 &= \|A(\vec{e}_i + \vec{e}_j)\|^2 = \|A\vec{e}_i + A\vec{e}_j\|^2 \\ &= (A\vec{e}_i + A\vec{e}_j) \cdot (A\vec{e}_i + A\vec{e}_j) \\ &= (A\vec{e}_i) \cdot (A\vec{e}_i) + 2(A\vec{e}_i) \cdot (A\vec{e}_j) + (A\vec{e}_j) \cdot (A\vec{e}_j) \\ &= \|A\vec{e}_i\|^2 + 2(A\vec{e}_i) \cdot (A\vec{e}_j) + \|A\vec{e}_j\|^2 \\ &= \|\vec{e}_i\|^2 + 2(A\vec{e}_i) \cdot (A\vec{e}_j) + \|\vec{e}_j\|^2 \\ &= 1 + 2(A\vec{e}_i) \cdot (A\vec{e}_j) + 1 \end{aligned}$$

Thus we see $2(A\vec{e}_i) \cdot (A\vec{e}_j) = 0$. Since $A\vec{e}_i$ and $A\vec{e}_j$ are exactly the i and j'th columns of A, this proves that any two distinct columns of A must be orthogonal.

Question 20. Suppose that A is a symmetric matrix, and that $\vec{v}$ and $\vec{w}$ are eigenvectors of A with eigenvalues λ and ω which are not the same. Show that $\vec{v} \perp \vec{w}$. (hint: think about $(A\vec{v}) \cdot \vec{w}$ and $\vec{v} \cdot (A\vec{w})$)

Solution 20. If A is symmetric, then $A = A^T$. Let $\vec{v}$ and $\vec{w}$ be eigenvectors of A with eigenvalues λ and ω, respectively, with $\lambda \neq \omega$. Then since $A^T = A$ and matrix multiplication is associative:

$$(A\vec{v}) \cdot \vec{w} = (A\vec{v})^T \vec{w} = (\vec{v}^T A^T)\vec{w} = \vec{v}^T (A^T \vec{w}) = \vec{v}^T (A\vec{w}) = \vec{v} \cdot (A\vec{w}).$$

Thus we see that $(A\vec{v}) \cdot \vec{w} = \vec{v} \cdot (A\vec{w})$. Now notice that

$$(A\vec{v}) \cdot \vec{w} = (\lambda\vec{v}) \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w}),$$

and also that

$$\vec{v} \cdot (A\vec{w}) = \vec{v} \cdot (\omega\vec{w}) = \omega(\vec{v} \cdot \vec{w})$$

Therefore we must have

$$\lambda(\vec{v} \cdot \vec{w}) = \omega(\vec{v} \cdot \vec{w}).$$

Since $\lambda \neq \omega$. This implies that $\vec{v} \cdot \vec{w} = 0$. Thus $\vec{v} \perp \vec{w}$.