**Math 308**
**Fall 2013**
**Midterm Exam**
**October 29, 2013**
**Time Limit: 50 Minutes**

**Name (Print):** ______________________

**Student ID:** ______________________

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This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 50 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | **??** | |
| Total: | 90 | |

1. (50 points) In each of the following, let $\mathcal{A} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ be a collection of $m$ vectors in $\mathbb{R}^n$, let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function, and let $A\vec{x} = \vec{b}$ be a linear system of equations

   (i) State the definition of $\mathcal{A}$ being linearly dependent

   (ii) State the definition of $f$ being one-to-one

   (iii) State the definition of the kernel of a linear transformation

   (iv) State the definition of the range of $f : \mathbb{R}^m \to \mathbb{R}^n$

   (v) State the definition of a linear combination of $\mathcal{A}$

   (vi) State the definition of a free variable of a system of equations

   (vii) State the definition of a spanning set for a spanning set of $\mathbb{R}^n$

   (viii) State the definition of $f$ being a linear transformation

   (ix) State the definition of the span span($\mathcal{A}$) of the set of vectors $\mathcal{A}$

   (x) State the definition of $A\vec{x} = \vec{b}$ being consistent

2. (10 points) Determine if each of the following linear systems of equations is consistent or inconsistent. Be sure to justify your reasoning. If the system is consistent, solve the system and write down the general solution in terms of free variables.

(i)

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ x_2 + x_3 &= 0 \\ 2x_3 &= 0 \end{aligned}$$

(ii)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ x_2 + x_3 &= 2 \\ x_1 + 3x_2 + 2x_3 &= 4 \end{aligned}$$

(iii)

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_5 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ x_3 + 2x_4 + x_5 &= 2 \end{aligned}$$

[10] Let $A$ be the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 & 5 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & 2 & 5 \end{bmatrix},$$

and let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear function defined by $f(\vec{x}) = A\vec{x}$.

(a) Find a linearly independent set of vectors which span $\ker(f)$

(b) Find a linearly independent set of vectors which span $\text{range}(f)$

(c) Determine all solutions to the linear system of equations

$$A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

(d) Determine all solutions to the linear system of equations

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

3. (10 points) Let

$$A = [\vec{a}_1 \vec{a}_2 \dots \vec{a}_m]$$

be an $n \times m$ matrix, with $\vec{a}_1, \dots, \vec{a}_m \in \mathbb{R}^n$ the column vectors of $A$. Suppose $c \in \mathbb{R}$ and

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ v_m \end{bmatrix}, \ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}.$$

(i) Prove that $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

(ii) Prove that $A(c\vec{u}) = cA(\vec{u})$

[Hint: remember that $A\vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + \cdots + u_m\vec{a}_m$ and a similar expression holds for $A\vec{v}$]

4. **Reasonable-Sounding Theorems Section!**

**Directions:** The "Reasonable Sounding Theorems Section" consists of several "theorems" proposed by excited linear algebra students. For each, determine whether the theorem is true or not. If it is true, write *TRUE*. Otherwise, write *FALSE*. It is not necessary to justify your answers for this question. Each part is worth 2pts:

(a) (2 points) If $V$ is a $d$-dimensional subspace of $\mathbb{R}^n$, and $\{\vec{v}_1, \ldots, \vec{v}_{d+1}\}$ is a set of vectors in $V$, then $\{\vec{v}_1, \ldots, \vec{v}_{d+1}\}$ is linearly dependent.

(b) (2 points) Any linear system of two unknowns and three equations must have infinitely many solutions

(c) (2 points) If $A$ is an $\ell \times m$ matrix and $B$ is an $m \times n$ matrix, and $A$ is nonsingular, then $\ker(AB) = \ker(B)$

(d) (2 points) If $A$ and $B$ are $n \times n$ square matrices, and $B = A^2 + 17A + 4I$, then $AB = BA$.

(e) (2 points) A linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ is onto if and only if it is one-to-one

5. (10 points)

   (a) Prove that if $n > m$, then a linear function $f : \mathbb{R}^m \to \mathbb{R}^n$ cannot be onto.

   (b) Prove that if $n < m$, then a linear function $f : \mathbb{R}^m \to \mathbb{R}^n$ cannot be one-to-one.

6. (10 points) (a) Give an example of two $2 \times 2$ matrices $A, B$ satisfying $AB \neq BA$

   (b) Give an example of a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ which is not linear

   (c) Give an example of a linear function $f : \mathbb{R}^3 \to \mathbb{R}^2$ which is onto but not one-to-one

   (d) Give an example of three linearly independent vectors in $\mathbb{R}^4$ that don't span $\mathbb{R}^4$

   (e) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is one-to-one but not onto.