Math 307 Quiz 2

March 2, 2015

Problem 1. Define what it means for the set of vectors $\{\vec{v}_1, \dots, \vec{v}_r\}$ to be linearly independent.

Solution 1. The set of vectors is linearly independent if the only linear combination giving the zero vector is the trivial linear combination.

Problem 2. Find all values of c for which the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} c \\ 8 \\ 9 \end{bmatrix} \right\}$$

is linearly dependent.

Solution 2. We set up the associated matrix

$$A = \begin{bmatrix} 1 & 4 & c \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix},$$

which identifies linear combinations of the original three vectors giving us the zero vector, and solutions to the homogeneous linear system of equations $A\vec{x} = \vec{0}$. Row reducing A, we get

$$B = \begin{bmatrix} 1 & 4 & c \\ 0 & -3 & 8-2c \\ 0 & 0 & -7+c \end{bmatrix}.$$

Thus $A\vec{x} = \vec{0}$ has more than the trivial solution if and only if $B\vec{x} = \vec{0}$ has more than the trivial solution, ie. if and only if it has a free variable – this is exactly the case when $-7+c=0$. Therefore the set of vectors is linearly dependent exactly when $c = 7$.

Problem 3. Give an example of a set of vectors in $\mathbb{R}^4$ which spans $\mathbb{R}^4$ but is not linearly independent.

Solution 3. There are lots of examples. However, any such example must be a set with at least five vectors. One example is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Problem 4. Give an example of a set of vectors in $\mathbb{R}^4$ which are linearly independent but do not span all of $\mathbb{R}^4$.

Solution 4. There are lots of examples. However, any such example must consist of no more than three vectors. One example is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$