Math 309 Section F **Name (Print):** ______________
Fall 2015
Final **Student ID:** ______________
December 17, 2015
Time Limit: 1 Hour 50 Minutes

This exam contains 12 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	15	
4	20	
5	20	
6	10	
7	15	
8	10	
9	10	
Total:	120	

1. (10 points) Find the general solution of the equation

$$\frac{d}{dt}\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

Solution 1. The characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ is $x^2 - 2x - 3 = (x-3)(x+1)$. Therefore the eigenvalues are -1 and 3. The corresponding eigenspaces are then easily identified to be:

$$E_{-1}(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}, \quad E_3(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Now to calculate the general solution, we must find a particular solution. For this, we have three different possible methods.

Method 1: Undetermined coefficients

We propose a solution of the form $\vec{y}_p(t) = \vec{c}e^t$. Plugging this into the system, we obtain $\vec{c}e^t = A\vec{c}e^t + \binom{2}{-1}e^t$, and therefore $(I - A)\vec{c}e^t = \binom{2}{-1}e^t$. It follows that

$$\vec{c} = (I - A)^{-1}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & -1/4 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix}$$

This gives us $\vec{y}_p = \binom{1/4}{-2}e^t$.

Method 2: Diagonalization

If we let P be the matrix whose columns are an eigenbasis for A, eg. $P = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$, then we have that $P^{-1}AP = D$ for the diagonal matrix $D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$. Now with this in mind, we define a new function $\vec{z}(t) = P^{-1}\vec{y}(t)$. In terms of $\vec{z}(t)$ our original differential equation becomes:

$$\frac{d}{dt}P\vec{z}(t) = AP\vec{z}(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

Multiplying both sides by P^{-1}, and using the fact that $P^{-1}AP = D$, we then obtain

$$\frac{d}{dt}\vec{z}(t) = D\vec{z}(t) + P^{-1}\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

Now we calculate that

$$P^{-1}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -5/4 \\ 3/4 \end{pmatrix}.$$

Therefore we are trying to solve

$$\frac{d}{dt}\vec{z}(t) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\vec{z}(t) + \begin{pmatrix} -5/4 \\ 3/4 \end{pmatrix} e^t.$$

Since everyting is now diagonal, a particular solution is calculated in the usual Math 307 way.

We get $\vec{z}_p(t) = \binom{-5/8}{-3/8} e^t$, and therefore

$$\vec{y}_p(t) = P\vec{z}_p(t) = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t.$$

Method 3: Variation of Parameters
Note that since we have an eigenbasis for A, the functions $\binom{-1}{2} e^{-t}$ and $\binom{1}{2} e^{3t}$ form a fundamental set of solutions to the corresponding homogeneous equation. Therefore we have a fundamental matrix

$$\Psi(t) = \begin{pmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{pmatrix}.$$

Note this is note the same as $\exp(At)$, but that's okay – there's more than one fundamental matrix!! Now the method of variation of parameters gives us the solution

$$\vec{y}_p = \Psi(t) \int \Psi(t)^{-1} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t dt.$$

Now we should be careful here too: since we did not choose the matrix exponential as our fundamental matrix, it is NOT true that $\Psi(-t) = \Psi(t)^{-1}$. We actually have to calculate the inverse directly:

$$\Psi(t)^{-1} = \begin{pmatrix} (-1/2)e^{t} & (1/4)e^{t} \\ (1/2)e^{-3t} & (1/4)e^{-3t} \end{pmatrix}$$

Trhowing everything together and doing the appropriate integral, we then see that

$$\vec{y}_p(t) = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t.$$

Regardless of the method, to get the general solution, we put together the general solution to the corresponding homogeneous equation with the particular solution:

$$\vec{y}(t) = \vec{y}_p(t) + \vec{y}_h(t) = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t + c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

where c_1 anc c_2 are arbitrary constants. Alternatively, we can write

$$\vec{y}(t) = \vec{y}_p(t) + \vec{y}_h(t) = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t + \begin{pmatrix} -e^{-t} & e^{3t} \\ 2e^{-t} & 2e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

2. (a) (5 points) Find the solution of the heat equation problem

$$u_t - 3u_{xx} = 0, \quad u(0,t) = 0, \ u(4,t) = 0, \quad u(x,0) = \sin(\pi x) - 2\sin(3\pi x/4).$$

(b) (5 points) Find the solution of the heat equation problem

$$u_t - 5u_{xx} = 0, \quad u_x(0,t) = 0, \ u_x(6,t) = 0, \quad u(x,0) = 4 - 2\cos(\pi x/3) + 7\cos(3\pi x/2).$$

Solution 2. On this problem, if you did an integral at any point, you wasted effort. You should be able to do this without an integral, because the initial condition is already written in terms of an appropriate linear combination of trig functions.

(a) In this case $L = 4, \alpha^2 = 3$, and the boundary conditions are homogeneous Dirichlet, and therefore we expect

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2\pi^2 t/16} \sin(n\pi x/4).$$

Plugging in $t = 0$, this says

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/4),$$

and since $u(x,0) = \sin(\pi x) - 2\sin(3\pi x/4)$, by inspection, we see that we should take $b_4 = 1, b_3 = -2$,a nd $b_n = 0$ otherwise. Thus

$$u(x,t) = e^{-3\pi^2 t} \sin(\pi x) - 2e^{-27\pi^2 t/16} \sin(3\pi x/4).$$

(b) In this case $L = 6, \alpha^2 = 5$, and the boundary conditions are homogeneous von Neumann, and therefore we expect

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-5n^2\pi^2 t/36} \cos(n\pi x/6).$$

Plugging in $t = 0$, this says

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/6),$$

and since $4 - 2\cos(\pi x/3) + 7\cos(3\pi x/2)$, by inspection, we see that we should take $a_0 = 8, a_2 = -2, a_9 = 7$, and $a_n = 0$ otherwise. Thus

$$u(x,t) = 4 - 2e^{-5\pi^2 t/9} \cos(\pi x/3) + 7e^{-45\pi^2 t/4} \cos(3\pi x/2).$$

3. (15 points) Consider the wave equation problem

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(0,t) = 0, u(L,t) = 0,\ t > 0$$

$$u(x,0) = f(x), u_t(x,0) = 0,\ 0 \le x \le 7$$

where $L = 7$, $c = 1$, and

$$f(x) = \begin{cases} 0, & 0 \le x < 3 \\ 1, & 3 \le x < 4 \\ 0, & 4 \le x < 7 \end{cases}$$

(a) Find the Fourier series solution $u(x,t)$ of the above wave equation for $0 \le x \le 7$ and $t > 0$ (eg. do not use D'Alembert)

(b) Sketch a graph of the odd, $2L$-periodic extension of $f(x)$, including at least two full periods

(c) Sketch a graph of the solution $u(x,t)$ when $t = 4$ (for this you should use D'Alembert).

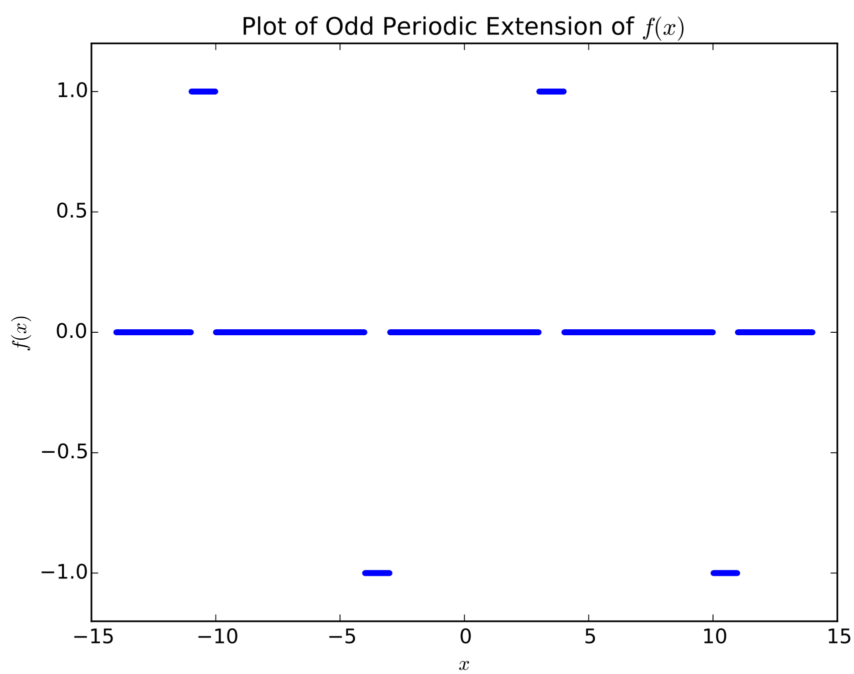
Solution 3.

(a) To do this, first note that $f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$ for $0 \le x \le L$ and

$$a_n = \int_0^L f(x)\sin(n\pi x/L) = \int_3^4 \sin(n\pi x/7) = \frac{-7}{n\pi}(\cos(4n\pi/7) - \cos(3n\pi/7)).$$

Therefore the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-7}{n\pi}(\cos(4n\pi/7) - \cos(3n\pi/7))\sin(n\pi x/7)\cos(n\pi t/7).$$



(b)

(c) From D'Alembert's solution, if we extend the definition of $f(x)$ to be defined on the entire real line by first letting f be odd in the interval $[-7, 7]$, and then be 14-periodic, as in the picture in (b), then since $c = 1$ the solution is given by

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right).$$

Now when $t = 4$, we have that

$$f(x+4) = \begin{cases} 0, & 0 \le x < 3 \\ 0, & 3 \le x < 6 \\ -1, & 6 \le x < 7 \end{cases}$$

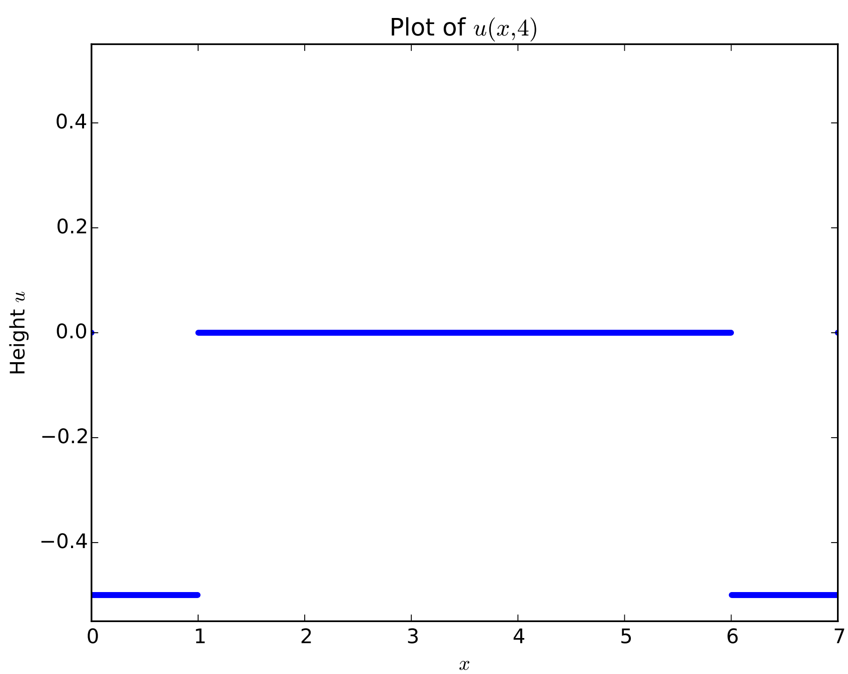
and also that

$$f(x-4) = \begin{cases} -1, & 0 \le x < 1 \\ 0, & 1 \le x < 4 \\ 0, & 4 \le x < 7 \end{cases}$$

Therefore

$$u(x,4) = \begin{cases} -1/2, & 0 \le x < 1 \\ 0, & 1 \le x < 6 \\ -1/2, & 6 \le x < 7 \end{cases}$$

A plot is included below.



4. (20 points) Find the solution of Laplace's equation

$$u_{xx} + u_{yy} = 0,$$

inside the interior of the rectangle bounded by the lines $x = 0, x = 1, y = 0$, and $y = 2$, and satisfying the boundary conditions (for $0 \leq x \leq 1$ and $0 \leq y \leq 2$):

$$u(x, 0) = 0,\ u(x, 2) = 0,\ u(0, y) = 0,\ u(1, y) = 1 - |y - 1|.$$

Solution 4. We calculate the sine transform of the initial condition with period $2L = 4$:

$$u(1, y) = \sum_{n=1}^{\infty} \frac{8\sin(\pi n/2)}{n^2\pi^2} \sin(n\pi y/2).$$

Now the nonhomogeneous boundary is the East wall, and therefore we expect that the solution is of the form

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi x/2)\sin(n\pi y/2).$$

In particular, this says that

$$u(1, y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi/2)\sin(n\pi y/2),$$

and from our sine transformation above, we know that we should take

$$c_n = \frac{8\sin(\pi n/2)}{n^2\pi^2\sinh(n\pi/2)}.$$

Therefore we find

$$u(x, y) = \sum_{n=1}^{\infty} \frac{8\sin(\pi n/2)}{n^2\pi^2\sinh(n\pi/2)} \sinh(n\pi x/2)\sin(n\pi y/2).$$

5. (20 points) Find the general solution of the system of equations

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t), \quad A = \begin{pmatrix} 3 & -2 & 2 \\ -6 & 7 & 2 \\ -6 & 6 & 3 \end{pmatrix}.$$

Solution 5. The first thing that we do is calculate the eigenvalues of the matrix A. To do this, we will use elementary row operations to create a shortcut. We want to know

$$\det(A - xI) = \det \begin{pmatrix} 3-x & -2 & 2 \\ -6 & 7-x & 2 \\ -6 & 6 & 3-x \end{pmatrix}.$$

Now recall that adding a multiple of one row to another, or one column to another, does not change the determinant! Therefore

$$\det(A - xI) = \det \begin{pmatrix} 3-x & -2 & 2 \\ 0 & 1-x & -1+x \\ -6 & 6 & 3-x \end{pmatrix} = \det \begin{pmatrix} 3-x & 0 & 0 \\ 0 & 1-x & -1+x \\ -6 & 6 & 3-x \end{pmatrix}$$

From this we see that

$$\det(A - xI) = (3-x)(1-x)(9-x).$$

Therefore the eigenvalues are $1, 3, 9$. Now at this stage, we can go two routes – we can calculate eigenvectors for each eigenvalue, or we can just calculate the matrix exponential using Sylvester's formula. Let's go the first route. We calculate

$$E_1(A) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}, \quad E_3(A) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}, \quad E_9(A) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Thus the general solution is

$$\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{9t}.$$

6. (10 points) Use separation of variables to convert the PDE

$$u_{xx} + u_{xt} + u_t = 0$$

into two second-order ODEs.

Solution 6. We assume $u(x,t) = F(x)G(t)$, so that

$$F''(x)G(t) + F'(x)G'(t) + F(x)G'(t) = 0.$$

Now we can rewrite this as

$$F''(x)G(t) = -(F'(x) + F(x))G'(t),$$

and dividing by $G(t)(F'(x) + F(x))$, we obtain:

$$F''(x)/(F'(x) + F(x)) = -G'(t)/G(t).$$

Now this says that a function of x only is equal to a function of t only, and so both must be equal to a constant $-\lambda$. In other words

$$F''(x)/(F'(x) + F(x)) = -\lambda, \quad -G'(t)/G(t) = -\lambda.$$

This simplifies to

$$F''(x) + \lambda F'(x) + \lambda F(x) = 0, \quad G'(t) - \lambda G(t) = 0.$$

7. (15 points) Calculate the Fourier series of the function

$$f(x) = x^2 - x, \ -1 \leq x \leq 1$$

with $f(x+2) = f(x)$ for all x.

Solution 7. Using the Euler-Fourier formulas, and even-odd arguments:

$$a_n = \frac{1}{1}\int_{-1}^{1}(x^2 - x)\cos(n\pi x)dx = \frac{1}{1}\int_{-1}^{1} x^2\cos(n\pi x)dx = \frac{4(-1)^n}{n^2\pi^2},$$

except for a_0, which is given by $2/3$. Furthermore,

$$b_n = \frac{1}{1}\int_{-1}^{1}(x^2 - x)\sin(n\pi x)dx = \frac{1}{1}\int_{-1}^{1} -x\sin(n\pi x)dx = \frac{2(-1)^n}{n\pi}.$$

Therefore we find

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n\pi}\sin(n\pi x) + \frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x)\right).$$

8. (10 points) Find the solution of the heat equation problem

$$u_t - 3u_{xx} = 0, \quad u(0,t) = 3,\ u(1,t) = 1, \quad u(x,0) = -3(x^2-1) + 3x$$

Solution 8. This is a problem with nonhomogeneous boundary conditions, so we first need to calculate the steady state solution. Doing this the usual way, we obtain

$$u_{\text{steady}}(x) = -2x + 3.$$

Then we define $v(x,t) = u(x,t) - u_{\text{steady}}(x)$, which satisfies

$$v_t - 3v_{xx} = 0, \quad v(0,t) = 0,\ v(1,t) = 0, \quad v(x,0) = -3x^2 + 5x.$$

This is a heat equation problem with homogeneous Dirichlet boundary conditions, and therefore since $\alpha^2 = 3, L = 1$ we know that

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2\pi^2 t} \sin(n\pi x).$$

The values of the b_n are given by the appropriate sine series expansion of $v(x,0)$, eg.

$$b_n = 2\int_0^1 (-3x^2 + 5x)\sin(n\pi x)dx = \frac{-4}{n^3\pi^3}((n^2\pi^2 + 3)(-1)^n - 3).$$

Thus

$$u(x,t) = v(x,t) + u_{\text{steady}}(x) = -2x + 3 + \sum_{n=1}^{\infty} \frac{-4}{n^3\pi^3}((n^2\pi^2+3)(-1)^n - 3)e^{-3n^2\pi^2 t}\sin(n\pi x).$$

9. (10 points) Find a solution to the wave equation problem

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(0,t) = 0, u(L,t) = 0,\ t > 0$$

$$u(x,0) = 0,\ u_t(x,0) = g(x),\ 0 \le x \le 1$$

where $L = 1$, $c = 2$, and $g(x) = x$ for $0 \le x \le 1$.

Solution 9. First of all, we look at the 2-periodic sine series for $g(x)$, which is

$$g(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin(n\pi x).$$

Then the boundary conditions and the initial conditions tell us that the solution we are looking for should be of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \sin(2n\pi t).$$

We calculate that

$$u_t(x,t) = \sum_{n=1}^{\infty} 2n\pi c_n \sin(n\pi x) \cos(2n\pi t),$$

and therefore

$$u_t(x,0) = \sum_{n=1}^{\infty} 2n\pi c_n \sin(n\pi x).$$

Thus from the above sine series, we should take $c_n = \frac{-(-1)^n}{n^2\pi^2}$, making

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2} \sin(n\pi x) \sin(2n\pi t).$$