Math 309 Section F
Fall 2015
Midterm
October 30, 2015
Time Limit: 50 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	10	
Total:	75	

Do not write in the table to the right.

1. (15 points) As quickly as you can, write down a *real* fundamental matrix for each of the following systems of equations.

   (a)

$$\begin{aligned} x' &= x - 4y \\ y' &= 4x - 7y \end{aligned}$$

   (b)

$$\begin{aligned} x' &= x - y \\ y' &= 5x - 3y \end{aligned}$$

   (c)

$$\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned}$$

2. (15 points) Werewolves and vampires do not prey on one-another, but do complete for the same limited food supply – the resident human population. In this way, increases in the vampire population lead to decreases in the human population and vise-versa. Let v and w be the populations of vampires and werewolves (in units of 1000) at time t. According to the empirical observations of Buffy the Vampire Slayer, the Winchester brothers, Bruce Campbell, Al Gore, and other experts, the qualitative behavior of the populations of each monster species is governed by the equations

$$v'(t) = v(1 - v - w)$$
$$w'(t) = w(0.75 - w - 0.5v)$$

(a) Determine the critical points of the above system.

(b) 0ne of the critical points you found in (a) should have been $(0.5, 0.5)$. Determine whether this critical point is stable, asymptotically stable, or unstable, and also whether it is a node, saddle, or spiral point. [Hint: you may want to use the Jacobian]

(c) Suppose that the initial population consists of 250 vampires and 500 werewolves. Qualitatively describe the populations at large time t

3. (15 points) Consider the matrix

$$A = \begin{pmatrix} 1 & -4 & -4 \\ -2 & 3 & 4 \\ 2 & -4 & -5 \end{pmatrix}$$

(a) determine the eigenvalues of A

(b) for each eigenvalue, calculate the corresponding eigenspace and determine its dimension. Is the matrix diagonalizable?

(c) calculate the value of A^{2015}

4. Provide an example of each of the following, if an example exists. If no example exists, write DOES NOT EXIST in big, bold text.

   (a) (2 points) Two *different* 3×3 matrices A and B, each with eigenvalues $1, -2, 4$

   (b) (2 points) A 3×3 matrix with a generalized eigenvector of rank 3 (just the matrix; you don't need to tell me a vector)

   (c) (2 points) A square matrix A whose matrix exponential $\exp(A)$ is not invertible

   (d) (2 points) A two-dimensional subspace of $\mathbb{R}^4$

   (e) (2 points) A linear system of algebraic equations with exactly two solutions.

5. (10 points) Find a solution to the differential equation

$$\frac{d}{dt}\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{y}(t).$$

satisfying the initial condition $x(0) = 3, y(0) = -1$.

6. (10 points) Find the general solution of the equation

$$\frac{d}{dt}\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$