Math 309 Section F
Fall 2015
Midterm
October 27, 2015
Time Limit: 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	10	
6	15	
Total:	75	

1. (15 points) Calculate the matrix exponential $\exp(At)$ for each of the following values of A. You may calculate this any way you like, as quickly as you can. Simplify your answer as much as possible.

   (a)

$$A = \begin{pmatrix} 7 & -1 \\ 25 & -3 \end{pmatrix}$$

   (b)

$$A = \begin{pmatrix} 27 & -8 \\ 84 & -25 \end{pmatrix}$$

   (c)

$$A = \begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix}$$

2. (10 points) A certain species of lizards, known as Rock-Paper-Scissors lizards exhibit interesting population behavior. A population of Rock-Paper-Scissors lizards has males of three different kinds: big, slow yellow-throated lizards; medium sized, blue-throated lizards; and small, quick red-throated lizards. Each color of male Rock-Paper-Scissors lizard uses its own special skills to mate with as many females as possible, and to prevent competing males from mating. Yellow-throated lizards can keep blue-throated lizards from mating, but are too slow to catch red-throated lizards. Blue-throated lizards are fast enough to keep the red-throated lizards from mating, but too small to compete with their yellow-thorated cousins. Red-throated lizards can quickly mate before yellow-throated lizards have the chance, but are easily deterred by blue-throats. In this way, the current population of any color of lizard affects the rate of change of other populations.

   Let $y(t), b(t), r(t)$ denote, respectively, the fraction of the total male population of yellow, blue, and red-throated lizards as a function of time (so that $y(t) + b(t) + r(t) = 1$). A simple linear model describing the strange interplay between these populations is

$$\begin{aligned} y'(t) &= (2/3)b(t) - (2/3)r(t) \\ b'(t) &= -(2/3)y(t) + (2/3)r(t) \\ r'(t) &= (2/3)y(t) - (2/3)b(t) \end{aligned}$$

   (a) substitute $y = 1 - r - b$ to reduce this to a two-dimension nonhomogeneous linear system describing $b(t)$ and $r(t)$

   (b) determine and classify the critical point(s) of this new system (are they stable, asymptotically stable, unstable?)

   (c) describe the behavior of the functions $y(t), b(t)$ and $r(t)$ as $t \to \infty$. What does this say about the diversity of the Rock-Paper-Scissors lizard population at large time?

3. For each of the following statements, write TRUE if the statement is TRUE, and FALSE if the statement is false. If the statement is false, also provide a counter-example.

   (a) (2 points) All 3×3 matrices are diagonalizable.

   (b) (2 points) If $\Psi(t)$ is a fundamental matrix for the equation $\vec{y}'(t) = A(t)\vec{y}(t)$ on the interval (α, β), then $\Psi(t)$ is invertible everywhere on the interval (α, β)

   (c) (2 points) If A and B are any two 2×2 matrices, then $e^{At}e^{Bt} = e^{(A+B)t}$

   (d) (2 points) If A and B are two similar square matrices, then they have the same determinant

   (e) (2 points) If A is a 2×2 matrix, and one of its eigenvalues is i, then $A^2 = -I$

4. (a) (5 points) Let $V \subseteq \mathbb{R}^n$ be a vector space. Define what it means for a collection of vectors to be a basis for V. How is the dimension of a vector space defined?

(b) (5 points) Let A be an $n \times n$ matrix. Define what it means for $\vec{v}$ to be a rank r generalized eigenvector with eigenvalue λ. Are eigenvectors generalized eigenvectors?

(c) (5 points) Let A be an $n \times n$ matrix. Write down the definition of the geometric multiplicity of an eigenvalue λ of A. How does it compare to the algebraic multiplicity?

5. (10 points) Find a solution to the differential equation

$$\begin{aligned} x'(t) &= 3x(t) + y(t) \\ y'(t) &= -5x(t) - y(t) \end{aligned}$$

satisfying the initial condition $x(0) = 3, y(0) = -1$.

6. (15 points) Find a particular solution of the equation for $t > 0$

$$x'(t) = 2x(t) + 4y(t) - 2t$$
$$y'(t) = -x(t) - 2y(t) + t$$