Math 309 Section F
Fall 2015
Midterm
October 27, 2015
Time Limit: 50 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 13 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	10	
6	15	
Total:	75	

1. (15 points) Calculate the matrix exponential $\exp(At)$ for each of the following values of A. You may calculate this any way you like, as quickly as you can. Simplify your answer as much as possible.

(a)

$$A = \begin{pmatrix} 7 & -1 \\ 25 & -3 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 27 & -8 \\ 84 & -25 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix}$$

Solution 1.

(a) The eigenvalues of A are 2, repeated twice. Therefore the matrix exponential is given by

$$\exp(At) = e^{2t}(I + (A - 2I)t) = \begin{pmatrix} e^{2t}(1+5t) & -te^{2t} \\ 25te^{2t} & e^{2t}(1-5t) \end{pmatrix}$$

(b) The eigenvalues of A are 3 and -1. Therefore the matrix exponential is

$$\exp(At) = \frac{1}{4}e^{3t}(A - (-1)I) - \frac{1}{4}e^{-t}(A - 3I) = \begin{pmatrix} 7e^{3t} - 6e^{-t} & -2e^{3t} + 2e^{-t} \\ 21e^{3t} - 21e^{-t} & -6e^{3t} + 7e^{-t} \end{pmatrix}$$

(c) The eigenvalues of A are $1 \pm i$, and therefore the matrix exponential is

$$\exp(At) = e^{t}\cos(t) + (A - I)e^{t}\sin(t) = \begin{pmatrix} e^{t}\cos(t) + 2e^{t}\sin(t) & e^{t}\sin(t) \\ -5e^{t}\sin(t) & e^{t}\cos(t) - 2e^{t}\sin(t) \end{pmatrix}$$

2. (10 points) A certain species of lizards, known as Rock-Paper-Scissors lizards exhibit interesting population behavior. A population of Rock-Paper-Scissors lizards has males of three different kinds: big, slow yellow-throated lizards; medium sized, blue-throated lizards; and small, quick red-throated lizards. Each color of male Rock-Paper-Scissors lizard uses its own special skills to mate with as many females as possible, and to prevent competing males from mating. Yellow-throated lizards can keep blue-throated lizards from mating, but are too slow to catch red-throated lizards. Blue-throated lizards are fast enough to keep the red-throated lizards from mating, but too small to compete with their yellow-thorated cousins. Red-throated lizards can quickly mate before yellow-throated lizards have the chance, but are easily deterred by blue-throats. In this way, the current population of any color of lizard affects the rate of change of other populations.

   Let $y(t), b(t), r(t)$ denote, respectively, the fraction of the total male population of yellow, blue, and red-throated lizards as a function of time (so that $y(t) + b(t) + r(t) = 1$). A simple linear model describing the strange interplay between these populations is

$$\begin{aligned} y'(t) &= (2/3)b(t) - (2/3)r(t) \\ b'(t) &= -(2/3)y(t) + (2/3)r(t) \\ r'(t) &= (2/3)y(t) - (2/3)b(t) \end{aligned}$$

   (a) substitute $y = 1 - r - b$ to reduce this to a two-dimension nonhomogeneous linear system describing $b(t)$ and $r(t)$

   (b) determine and classify the critical point(s) of this new system (are they stable, asymptotically stable, unstable?)

   (c) describe the behavior of the functions $y(t), b(t)$ and $r(t)$ as $t \to \infty$. What does this say about the diversity of the Rock-Paper-Scissors lizard population at large time?

Solution 2.

(a) We substitute $y = 1 - r - b$ to obtain

$$b'(t) = (2/3)b(t) + (4/3)r(t) - (2/3)$$
$$r'(t) = -(4/3)b(t) - (2/3)r(t) + (2/3)$$

(b) Recall that the critical points of a system

$$\begin{pmatrix} b' \\ r' \end{pmatrix} = \vec{F}\left(\begin{pmatrix} b \\ r \end{pmatrix}\right), \quad \vec{F}\left(\begin{pmatrix} b \\ r \end{pmatrix}\right) = \begin{pmatrix} u(b,r) \\ v(b,r) \end{pmatrix}$$

are values of $\binom{b}{r}$ such that $\vec{F}\left(\binom{b}{r}\right) = \vec{0}$. In other words, they are the values of b and r such that both $u(b,r) = 0$ and $v(b,r) = 0$. In our case $u(b,r) = (2/3)b + (4/3)r - (2/3)$, and $v(b,r) = (-4/3)b + (-2/3)r + (2/3)$, and so we find that there is exactly one critical point: when $r = 1/3$ and $b = 1/3$. To classify the type, we approximate $\vec{F}\left(\binom{b}{r}\right)$ linearly based at $\binom{1/3}{1/3}$. Then

$$\vec{F}\left(\begin{pmatrix} b \\ r \end{pmatrix}\right) \approx \vec{F}\left(\begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}\right) + J \cdot \begin{pmatrix} b - 1/3 \\ r - 1/3 \end{pmatrix},$$

where here J is the Jacobian of $\vec{F}$ at the point $\binom{1/3}{1/3}$, eg.

$$J = \begin{pmatrix} u_b(1/3,1/3) & u_r(1/3,1/3) \\ v_b(1/3,1/3) & v_r(1/3,1/3) \end{pmatrix} = \begin{pmatrix} 2/3 & 4/3 \\ -4/3 & -2/3 \end{pmatrix}$$

Therefore since $\vec{F}\left(\binom{1/3}{1/3}\right) = \vec{0}$, we have that

$$\vec{F}\left(\begin{pmatrix} b \\ r \end{pmatrix}\right) \approx \begin{pmatrix} 2/3 & 4/3 \\ -4/3 & -2/3 \end{pmatrix} \begin{pmatrix} b - 1/3 \\ r - 1/3 \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix determine the type of critical point it is. The eigenvalues are $\pm i2/\sqrt{3}$, so the critical point is a stable spiral.

(c) As $t \to \infty$, the value of $\binom{b}{t}$ continues to rotate in a circle around the equilibrium value $\binom{1/3}{1/3}$. The ratios $y(t), b(t), r(t)$ are all periodic, and alternating between when they are maximized and minimized. In particular, the population remains very diverse over time.

3. For each of the following statements, write TRUE if the statement is TRUE, and FALSE if the statement is false. If the statement is false, also provide a counter-example.

   (a) (2 points) All 3×3 matrices are diagonalizable.

   (b) (2 points) If $\Psi(t)$ is a fundamental matrix for the equation $\vec{y}'(t) = A(t)\vec{y}(t)$ on the interval (α, β), then $\Psi(t)$ is invertible everywhere on the interval (α, β)

   (c) (2 points) If A and B are any two 2×2 matrices, then $e^{At}e^{Bt} = e^{(A+B)t}$

   (d) (2 points) If A and B are two similar square matrices, then they have the same determinant

   (e) (2 points) If A is a 2×2 matrix, and one of its eigenvalues is i, then $A^2 = -I$

Solution 3.

(a) This is FALSE. An example is the Jordan block

$$J_3(-14) = \begin{pmatrix} -14 & 1 & 0 \\ 0 & -14 & 1 \\ 0 & 0 & -14 \end{pmatrix}$$

(b) This is TRUE.

(c) This is FALSE. In order for this to work, both A and B must commute! For example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

so that

$$\exp((A+B)t) = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$$

However,

$$\exp(At) = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}, \quad \exp(Bt) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

so that

$$\exp(At)\exp(Bt) = \begin{pmatrix} e^t & te^t \\ 0 & 1 \end{pmatrix}$$

(d) This is TRUE.

(e) This is TRUE.

4. (a) (5 points) Let $V \subseteq \mathbb{R}^n$ be a vector space. Define what it means for a collection of vectors to be a basis for V. How is the dimension of a vector space defined?

(b) (5 points) Let A be an $n \times n$ matrix. Define what it means for $\vec{v}$ to be a rank r generalized eigenvector with eigenvalue λ. Are eigenvectors generalized eigenvectors?

(c) (5 points) Let A be an $n \times n$ matrix. Write down the definition of the geometric multiplicity of an eigenvalue λ of A. How does it compare to the algebraic multiplicity?

Solution 4.

(a) A collection $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of vectors of V is a basis for V if the set $\mathcal{B}$ is linearly independent, and the span of the set $\mathcal{B}$ is V. The dimension of V is the number of elements in a basis for V.

(b) A rank r generalized eigenvector $\vec{v}$ of A with eigenvalue λ is a vector $\vec{v} \in \mathbb{R}^n$ which is in the nullspace of $(A - \lambda I)^r$ but NOT in the nullspace of $(A - \lambda I)^{r-1}$. Eigenvectors are the same thing as generalized eigenvectors of rank 1.

(c) The geometric multiplicity of an eigenvalue λ of A is the dimension of the eigenspace $E_\lambda(A)$. The geometric multiplicity is bounded above by the algebraic multiplicity, but not necessarily equal to.

5. (10 points) Find a solution to the differential equation

$$x'(t) = 3x(t) + y(t)$$
$$y'(t) = -5x(t) - y(t)$$

satisfying the initial condition $x(0) = 3, y(0) = -1$.

Solution 5. We rewrite this system in matrix form as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix}.$$

A fundamental matrix for this system is given by the matrix exponential $\Psi(t) = \exp(At)$. We already calculated this in Problem 1.c above. Thus we have a fundamental matrix

$$\Psi(t) = \begin{pmatrix} e^t \cos(t) + 2e^t \sin(t) & e^t \sin(t) \\ -5e^t \sin(t) & e^t \cos(t) - 2e^t \sin(t) \end{pmatrix}$$

The general solution to the equation is $\binom{x(t)}{y(t)} = \Psi(t) \cdot \vec{c}$. We just need to figure out which value of $\vec{c}$ gives us the appropriate initial condition. Note that $\Psi(0) = I$. Therefore plugging in 0, we find

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \Psi(0) \cdot \vec{c} = I \cdot \vec{c} = \vec{c}.$$

Therefore we should take $\vec{c} = \binom{3}{-1}$. This tells us

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \Psi(t) \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3e^t \cos(t) + 5e^t \sin(t) \\ -e^t \cos(t) + 13e^t \sin(t) \end{pmatrix}.$$

6. (15 points) Find a particular solution of the equation for $t > 0$

$$x'(t) = 2x(t) + 4y(t) - 2t$$
$$y'(t) = -x(t) - 2y(t) + t$$

Solution 6. In terms of matrices, we may rewrite this as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} -2t \\ t \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}.$$

The eigenvalues of A are 0, repeated twice. Therefore a fundamental matrix is given by

$$\Psi(t) = \exp(At) = I + At = \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix}.$$

We can obtain from this a general solution by using the method of variation of parameters. We obtain

$$\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \Psi(t) \int \Psi(t)^{-1} \begin{pmatrix} -2t \\ t \end{pmatrix} dt \\
&= \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} \int \begin{pmatrix} 1-2t & -4t \\ t & 1+2t \end{pmatrix} \begin{pmatrix} -2t \\ t \end{pmatrix} dt \\
&= \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} \int \begin{pmatrix} -2t \\ t \end{pmatrix} dt \\
&= \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} \int \begin{pmatrix} -t^2 \\ (1/2)t^2 \end{pmatrix} dt = \begin{pmatrix} -t^2 \\ (1/2)t^2 \end{pmatrix}.
\end{aligned}$$