MATH 309: Homework #3

Due on: November 9, 2015

Problem 1 *Boundary Value Problems*

For each of the following boundary value problems, find all solutions to the boundary value problem or show that no solution exists.

(a) $y'' + y = 0$, $y(0) = 0$, $y'(\pi) = 1$

(b) $y'' + y = 0$, $y(0) = 0$, $y(L) = 0$

(c) $y'' + y = x$, $y(0) = 0$, $y(\pi) = 0$

. . . .

Solution 1. In each case, the general solution is

$$y(x) = A\cos(x) + B\sin(x),$$

so the question is whether or not we can find constants A, B satisfying the boundary conditions.

(a) The condition $y(0) = 0$ implies that $A = 0$. Therefore $y(x) = B\sin(x)$. The condition $y'(\pi) = 0$ implies that $B = 0$, and therefore the only solution is the trivial solution $y = 0$.

(b) The condition $y(0) = 0$ implies that $A = 0$. Therefore $y(x) = B\sin(x)$. The condition $y(L) = 0$ implies that $B\sin(L) = 0$, and therefore either $B = 0$, giving us the trivial solution, or else $L = n\pi$ for some integer n, in which case B can be anything! Thus we have two cases: if L is not an integer multiple of π, then the only solution is the trivial solution $y = 0$. If $L = n\pi$ for some integer n, then the family of all solutions is $y = B\sin(x)$.

(c) The condition $y(0) = 0$ implies that $A = 0$. Therefore $y(x) = B\sin(x)$, therefore the condition $y(\pi) = 0$ is automatically satisfied, leaving implies that $B = 0$, and therefore the only solution is the trivial solution $y = 0$.

Problem 2 *Dirichlet Eigenvalue Problem*

Determine for which values of λ the boundary value problem

$$y'' + \lambda y = 0, \; y(0) = 0, \; y(L) = 0,$$

has a solution and describe the solutions.

. . . .

Solution 2. It's important to note that the values of λ which work will be dependent on the value of L – this relationship between λ and L becomes important in the method of separation of variables later on. Let's first think about the general solution to $y'' + \lambda y$. The characteristic polynomial of this equation is $x^2 + \lambda$, which has roots $\pm\sqrt{-\lambda}$. The general solution therefore takes three distinct forms, depending on whether λ is positive, negative, or zero.

Case A ($\lambda < 0$):
In this case, $\sqrt{-\lambda}$ is real, so the general solution is

$$y = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

Then since $y(0) = 0$, we have $A + B = 0$. Furthermore, since $y(L) = 0$ we have $Ae^{\sqrt{-\lambda}L} + Be^{-\sqrt{-\lambda}L} = 0$. Thus we have a homogeneous system of two equations and two unknowns. In matrix form, this is

$$\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}L} & e^{-\sqrt{-\lambda}L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the above matrix is $e^{-\sqrt{-\lambda}L} - e^{\sqrt{-\lambda}L}$, which is nonzero. Therefore the matrix is nonsingular, and the homogeneous system of equations has exactly one solution: the trivial solution. Therefore $A = B = 0$, making $y = 0$ the only solution to the boundary value problem.

Case B ($\lambda = 0$):
In this case, $\sqrt{-\lambda}$ is 0, so the general solution is

$$y = A + Bx.$$

Then since $y(0) = 0$, we have $A = 0$. Furthermore, since $y(L) = 0$ we have $A + BL = 0$. Since $A = 0$, this also says that $B = 0$, and therefore the only solution is the trivial solution $y = 0$.

Case C ($\lambda > 0$):
In this case, $\sqrt{-\lambda} = i\sqrt{\lambda}$ is imaginary, so the general solution is

$$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Then since $y(0) = 0$, we have $A = 0$, making $y = B\sin(\sqrt{\lambda}x)$. Then since $y(L) = 0$, we have that $B = 0$ or $\sin(\sqrt{\lambda}L) = 0$. In the former case, $y = 0$. In the latter

case, $\sqrt{\lambda}L = n\pi$ for some integer n and therefore $\lambda = n^2\pi^2/L^2$. In this case $y = B\sin(\sqrt{\lambda}x) = B\sin(n\pi x/L)$ is a solution for any value of B.
SUMMARY:
The boundary value problem has at least one solution for every value of λ: the trivial solution. The boundary value problem has more than the trivial solution exactly when $\lambda = n^2\pi^2/L^2$ for some nonzero integer n, in which case anything of the form $B\sin(n\pi x/L)$ is a solution.

Problem 3 *von Neumann Eigenvalue Problem*

Determine for which values of λ the boundary value problem

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y'(L) = 0,$$

has a solution and describe the solutions.

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Solution 3. It's important to note that the values of λ which work will be dependent on the value of L – this relationship between λ and L becomes important in the method of separation of variables later on. Let's first think about the general solution to $y'' + \lambda y$. The characteristic polynomial of this equation is $x^2 + \lambda$, which has roots $\pm\sqrt{-\lambda}$. The general solution therefore takes three distinct forms, depending on whether λ is positive, negative, or zero.
Case A ($\lambda < 0$):
In this case, $\sqrt{-\lambda}$ is real, so the general solution is

$$y = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

We note that

$$y' = \sqrt{-\lambda}(Ae^{\sqrt{-\lambda}x} - Be^{-\sqrt{-\lambda}x}).$$

Then since $y'(0) = 0$, we have $A - B = 0$. Furthermore, since $y'(L) = 0$ we have $Ae^{\sqrt{-\lambda}L} - Be^{-\sqrt{-\lambda}L} = 0$. Thus we have a homogeneous system of two equations and two unknowns. In matrix form, this is

$$\begin{pmatrix} 1 & -1 \\ e^{\sqrt{-\lambda}L} & -e^{-\sqrt{-\lambda}L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the above matrix is $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}$, which is nonzero. Therefore the matrix is nonsingular, and the homogeneous system of equations has exactly one solution: the trivial solution. Therefore $A = B = 0$, making $y = 0$ the only solution to the boundary value problem.
Case B ($\lambda = 0$):
In this case, $\sqrt{-\lambda}$ is 0, so the general solution is

$$y = A + Bx.$$

We note that

$$y' = B$$

Then since $y'(0) = 0$, we have $B = 0$. Furthermore, since $y'(L) = 0$ we have $B = 0$, again. Thus $y = A$ is a solution for any value of A. **Case C** ($\lambda > 0$):
In this case, $\sqrt{-\lambda} = i\sqrt{\lambda}$ is imaginary, so the general solution is

$$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

We note that

$$y' = \sqrt{\lambda x}(B\cos(\sqrt{\lambda}x) - A\sin(\sqrt{\lambda}x)).$$

Then since $y'(0) = 0$, we have $B = 0$, making $y = A\cos(\sqrt{\lambda}x)$. Then since $y'(L) = 0$, we have that $A = 0$ or $\sin(\sqrt{\lambda}L) = 0$. In the former case, $y = 0$. In the latter case, $\sqrt{\lambda}L = n\pi$ for some integer n and therefore $\lambda = n^2\pi^2/L^2$. In this case $y = A\cos(\sqrt{\lambda}x) = A\cos(n\pi x/L)$ is a solution for any value of B.
SUMMARY:
The boundary value problem has at least one solution for every value of λ: the trivial solution. The boundary value problem has more than the trivial solution exactly when $\lambda = 0$ or $\lambda = n^2\pi^2/L^2$ for some nonzero integer n. If $\lambda = 0$, then anything of the form $y = A$ is a solution. If $\lambda = n^2\pi^2/L^2$, then anything of the form $y = A\cos(n\pi x/L)$ is a solution.

Problem 4 *Fourier Series*

For each of the following functions, sketch a graph of the function and find the Fourier series

(a) $f(x) = \sin^3(x) + \cos^2(2x + 3)$

(b) $f(x) = -x, \ -L \leq x < L$ with $f(x + 2L) = f(x)$ for all x

(c) $f(x) = \begin{cases} x + 1, & -\pi \leq x < 0 \\ 1 - x, & 0 \leq x < \pi \end{cases}$ with $f(x + 2\pi) = f(x)$ for all x

. . . .

Solution 4. We will omit the sketches, as we assume that students are able to figure that part out.

(a) The idea of this first problem is to use a little bit of trigonometry to write $f(x)$ as a finite sum of sines and cosines. This is easier here than trying to apply the Euler-Fourier formula directly. The triggy-tricks that we will use are the following:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta).$$

$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi).$$
$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi).$$
$$\sin(\theta)\cos(\phi) = \frac{1}{2}\left(\sin(\theta+\phi)+\sin(\theta-\phi)\right).$$

For starters:

$$\begin{aligned}\cos^2(2x+3) &= (1/2)+(1/2)\cos(4x+6)\\ &= (1/2)+(1/2)[\cos(4x)\cos(6)-\sin(4x)\sin(6)]\\ &= (1/2)+\frac{1}{2}\cos(6)\cos(4x)-\frac{1}{2}\sin(6)\sin(4x).\end{aligned}$$

Moreover:

$$\begin{aligned}\sin^3(x) &= (1-\cos^2(x))\sin(x)\\ &= ((1/2)-(1/2)\cos(2x))\sin(x)\\ &= (1/2)\sin(x)-(1/2)\cos(2x)\sin(x)\\ &= (1/2)\sin(x)-(1/4)(\sin(3x)-\sin(x))\\ &= \frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x).\end{aligned}$$

Therefore we have that $a_0 = 1, b_1 = 3/4, b_3 = 1/4, a_4 = \cos(6)/2, b_4 = \sin(6)/2$ and a_i, b_j are zero otherwise. In other words

$$f(x) = (1/2)+\frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x)+\frac{1}{2}\cos(6)\cos(4x)-\frac{1}{2}\sin(6)\sin(4x).$$

(b) The second function is a "sawtooth" wave. Note that $f(x)$ is odd, forcing $a_n = 0$ for all n. Therefore we need only worry about the b_n's. For these, we apply the Euler-Fourier formula:

$$\begin{aligned}b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{1}{L}\int_{-L}^{L} -x\sin\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{1}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{2L}{n\pi}\cos(n\pi)-\frac{1}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{2L}{n\pi}\cos(n\pi)-\frac{1}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{2L}{n\pi}\cos(n\pi)-\frac{L}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)|_{-L}^{L} = \frac{2L}{n\pi}\cos(n\pi).\end{aligned}$$

Then since $\cos(n\pi) = (-1)^n$, we see that

$$f(x) = \sum_{n=1}^{\infty}\frac{2L(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right).$$

(c) The third function is a “triangular wave”. Note that $f(x)$ is even, forcing $b_n = 0$ for all n. Therefore we need only worry about the a_n's. For these, we apply the Euler-Fourier formula:

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} (1-x)\cos(nx)dx \\
&= \frac{2}{n\pi}(1-x)\sin(nx)|_0^\pi + \frac{2}{n\pi}\int_0^\pi \sin(nx)dx \\
&= \frac{2}{n\pi}\int_0^\pi \sin(nx)dx = -\frac{2}{n^2\pi}\cos(nx)|_0^\pi \\
&= -\frac{2}{n^2\pi}(\cos(n\pi)-1) = -\frac{2}{n^2\pi}((-1)^n-1).
\end{aligned}$$

In particular, $a_n = 0$ when n is even. Note that in the above calculation, we used the fact that $n \neq 0$. We need to do the case $n = 0$ separately:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos(0\pi x/L)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = -(\pi-2).$$

Putting this all together we have

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n-1}\cos((2n-1)x) \\
&= \frac{2-\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2}\cos((2n-1)x)
\end{aligned}$$

Problem 5 *Parseval's Identity*

Let $f(x)$ be a periodic function with fundamental period $2L$, and suppose that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

Using the fact that

$$\left\{\frac{1}{2}, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) : n = 0,1,2,\ldots, m = 1,2,3,\ldots\right\}$$

is a mutually orthogonal set of functions, prove Parseval's identity:

$$\frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

. . . .

Solution 5. This problem is easier to understand if we use the inner product notation

$$\langle g(x), h(x)\rangle = \int_{-L}^{L} g(x), h(x) dx.$$

Then using the linearity of the inner product, we have that

$$\begin{aligned}\int_{-L}^{L} f(x)^2 dx &= \langle f, f\rangle \\ &= \left\langle f(x), \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]\right\rangle \\ &= a_0\left\langle f(x), \frac{1}{2}\right\rangle + \sum_{n=1}^{\infty} a_n \left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{n=1}^{\infty} b_n \left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle.\end{aligned}$$

For fixed n, we calculate using orthogonality:

$$\begin{aligned}\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle &= \left\langle \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right], \cos\left(\frac{n\pi x}{L}\right)\right\rangle \\ &= \left\langle \frac{a_0}{2}, \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{m=1}^{\infty} a_m \left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{m=1}^{\infty} b_m \left\langle \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle \\ &= a_m \left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle = a_m L.\end{aligned}$$

A similar calculation also shows

$$\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = b_n L.$$

and that

$$\left\langle f(x), \frac{1}{2}\right\rangle = \frac{1}{2}a_0 L.$$

Therefore we see that

$$\begin{aligned}&\int_{-L}^{L} f(x)^2 dx \\ &= a_0\left\langle f(x), \frac{1}{2}\right\rangle + \sum_{n=1}^{\infty} a_n \left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle + \sum_{n=1}^{\infty} b_n \left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle \\ &= a_0(a_0 L/2) + \sum_{n=1}^{\infty} a_n(a_n L) + \sum_{n=1}^{\infty} b_n(b_n L) = \frac{1}{2}a_0^2 L + \sum_{n=1}^{\infty}\left(a_n^2 L + b_n^2 L\right)\end{aligned}$$

Dividing now by L gives us Parseval's identity.

Problem 6 *Parseval's Identity Application*

Use Parseval's identity and the Fourier series for the square wave function

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \text{ with } f(x+2) = f(x) \text{ for all } x$$

to obtain the value of the infinite sum

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

. . . .

Solution 6. We first calculate the Fourier series for the square wave function above using the Euler-Fourier formula. We calculate

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x) = \int_{0}^{1} \cos(n\pi x) = \frac{1}{n\pi}\sin(n\pi x)|_0^1 = 0.$$

The above calculation does not work when $n = 0$ however (since we divided by n). We have to do this separately:

$$a_0 = \int_{-1}^{1} f(x)dx = 1.$$

We also calculate the b_n's:

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x) = \int_{0}^{1} \sin(n\pi x) = -\frac{1}{n\pi}\cos(n\pi x)|_0^1 = -\frac{1}{n\pi}((-1)^n - 1).$$

This last expression is 0 if n is even and 1 if n is odd. Therefore we have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_{2n-1}\sin(n\pi x)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\sin(n\pi x).$$

Then since

$$\frac{1}{1}\int_{-1}^{1} f(x)^2 dx = 1,$$

Parseval's identity tells us that

$$1 = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2}.$$

Simplifying this a bit, it says

$$\frac{1}{2} = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2},$$

and therefore

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Note that the sum on the left is exactly the sum that we were trying to calculate, just indexed a bit differently. In fact if we reindex by setting $m = n - 1$, then the above expression becomes

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$