MATH 309: Homework #4

Due on: November 20, 2015

Problem 1 *Even and Odd Functions*

Prove that any function $f(x)$ may be expressed as a sum of two functions $f(x) = g(x) + h(x)$ with $g(x)$ even and $h(x)$ odd. [Hint: consider $f(x) + f(-x)$].

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Problem 2 *Even and Odd Functions*

Prove that the derivative of an even function is odd and that the derivative of an odd function is even.

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Problem 3 *Sine Series*

Consider the function

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \\ 2, & 2\pi < x < 3\pi \end{cases}$$

(a) Scketch a graph of $f(x)$

(b) By reflecting f appropriately, express f as a sine series.

(c) Plot three different partial sums of the sine series, clearly indicating the partial sums being plotted.

(d) Sketch a graph of the function to which the sine series converges for three periods.

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Problem 4 *Cosine Series*

Consider the function

$$f(x) = \begin{cases} x, 0 < x < \pi \\ 0, \pi < x < 2\pi \end{cases}$$

(a) Scketch a graph of $f(x)$

(b) By reflecting f appropriately, express f as a cosine series.

(c) Plot three different partial sums of the cosine series, clearly indicating the partial sums being plotted.

(d) Sketch a graph of the function to which the cosine series converges for three periods.

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Problem 5 *Heat Equation 1*

Find the solution of the heat conduction problem

$$\begin{aligned} 100u_{xx} &= u_t, \quad 0 < x < 1, \ t > 0 \\ u(0,t) &= u(1,t) = 0, \ t > 0 \\ u(x,0) &= \sin(2\pi x) - \sin(5\pi x) \end{aligned}$$

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Problem 6 *Heat Equation 2*

Find the solution of the heat conduction problem

$$\begin{aligned} u_{xx} &= 4u_t, \quad 0 < x < 2, \ t > 0 \\ u(0,t) &= u(2,t) = 0, \ t > 0 \\ u(x,0) &= 2\sin(\pi x/2) - \sin(\pi x) + 4\sin(2\pi x) \end{aligned}$$

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Problem 7 *Schrödinger Equation*

In quantum mechanics, the position of a point particle in space is not certain – it's described by a probability distribution. The probability distribution of the position of the particle is $|\psi(x,t)|^2$, where $\psi(x,t)$ is the **wave function** of the particle. (Note: the wave function $\psi(x,t)$ can be complex-valued!!). The one-dimensional, time-dependent

Schrödinger equation, describing the wave function $\psi(x,t)$ of a particle of mass m interacting with a potential $v(x)$ is given by

$$i\hbar\psi_t(x,t) = -\frac{\hbar^2}{2m}\psi_{xx}(x,t) + v(x)\psi(x,t)$$

where $\hbar$ is some universal constant. The potential $v(x)$ can be imagined as a function describing the particles interaction with whatever "stuff" is in the space surrounding the particle, eg. walls, external forces, etc.

(a) Use separation of variables to replace this partial differential equation with a pair of two ordinary differential equations

(b) If $v(x)$ is a potential corresponding to an "infinite square well":

$$v(x) = \begin{cases} 0, & -1 < x < 1 \\ \infty, & |x| \geq 1 \end{cases}$$

Then $\psi(x,t)$ must be zero whenever $|x| \geq 1$ and therefore $\psi(x,t)$ is the wave function of a particle trapped in a one-dimensional box! In other words, this potential describes a particle surrounded by impermeable walls. In this case, Schrödinger's equation reduces to

$$\begin{aligned} i\hbar\psi_t(x,t) &= -\frac{\hbar^2}{2m}\psi_{xx}(x,t), \quad -1 < x < 1,\ t > 0 \\ \psi(-1,t) &= \psi(1,t) = 0,\ t > 0 \end{aligned}$$

Suppose that initially the wave function is known to be

$$\psi(x,0) = \frac{3}{5}\sin(\pi x) + \frac{4}{5}\sin(3\pi x).$$

Determine $\psi(x,t)$ for all $t > 0$.

(c) Since $|\psi(x,t)|^2$ is the probability *distribution* of the particle's position at time t, the probability that the particle is somewhere in the box between ℓ_1 and ℓ_2 is given by

$$\mathbb{P}(\ell_1 \leq \text{pos} \leq \ell_2) = \int_{\ell_1}^{\ell_2} |\psi(x,t)|^2 dx.$$

Show that the probability $\mathbb{P}(-1 \leq \text{pos} \leq 1)$ that the particle is between -1 and 1 is always 1 (in other words, the particle is always in the box!).

(d) What is the probability $\mathbb{P}(-1 \leq \text{pos} \leq 0)$ that the particle is in the first half of the box at any given time?

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