MATH 309: Homework #4

Due on: November 20, 2015

Problem 1 *Even and Odd Functions*

Prove that any function $f(x)$ may be expressed as a sum of two functions $f(x) = g(x) + h(x)$ with $g(x)$ even and $h(x)$ odd. [Hint: consider $f(x) + f(-x)$].

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Solution 1. In order to prove the statement we want, we need to show that for any function $f(x)$, there exists an even function $g(x)$ and an odd function $h(x)$ with $f(x) = g(x) + h(x)$. In particular, we need to come up with equations for $g(x)$ and $h(x)$ in terms of $f(x)$. How can we do this? One way is to assume that $g(x)$ and $h(x)$ are known to exist, and then fiddle around with $f(x)$ to figure out the equations. In particular if $g(x)$ is even and $h(x)$ is odd and $f(x) = g(x) + h(x)$ then

$$f(-x) = g(-x) + h(-x) = g(x) - h(x).$$

It follows that

$$f(x) + f(-x) = g(x) + h(x) + (g(x) - h(x)) = 2g(x),$$

and therefore we should take $g(x) = (f(x) + f(-x))/2$. Similarly, we have that

$$f(x) - f(-x) = g(x) + h(x) - (g(x) - h(x)) = 2h(x),$$

and therfore we should take $h(x) = (f(x) - f(-x))/2$. Great!

What we did above is just a bunch of scratch work. Here's our actual proof: Suppose that $f(x)$ is a function. Define $g(x) = (f(x) + f(-x))/2$ and $h(x) = (f(x) - f(-x))/2$. Then since

$$g(-x) = (f(-x) + f(- - x))/2 = (f(-x) + f(x))/2 = (f(x) + f(-x))/2 = g(x)$$

we have that $g(x)$ is even. Similarly

$$h(-x) = (f(-x) - f(- - x))/2 = (f(-x) - f(x))/2 = -(f(x) - f(-x))/2 = -h(x)$$

and therefore $h(x)$ is odd. Furthermore

$$g(x) + h(x) = (f(x) + f(-x))/2 + (f(x) - f(-x))/2 = f(x).$$

Therefore $f(x) = g(x) + h(x)$ is a sum of an even function and an odd function. This completes our proof.

Problem 2 *Even and Odd Functions*

Prove that the derivative of an even function is odd and that the derivative of an odd function is even.

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Solution 2. There are many great ways to prove this fact. We will use one of the simplest methods: the chain rule. Let $g(x) = f(-x)$. Then by the chain rule

$$g'(x) = -f'(-x).$$

Now let's suppose $f(x)$ is an even function. Then in this case $g(x) = f(x)$, making $g'(x) = f'(x)$, so that the above expression reads $f'(x) = -f'(-x)$. Since x was arbitrary, this shows that $f'(x)$ is odd when $f(x)$ is even. Alternatively, let's suppose that $f(x)$ is an odd function. Then $g(x) = -f(x)$, making $g'(x) = -f'(x)$, so that the expression we derived from the chain rule reads $-f'(x) = -f'(-x)$, and hence $f'(x) = f'(-x)$. Since x was arbitrary, this shows that $f'(x)$ is even when $f(x)$ is odd. This completes our proof.

Problem 3 *Sine Series*

Consider the function

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \\ 2, & 2\pi < x < 3\pi \end{cases}$$

(a) Scketch a graph of $f(x)$

(b) By reflecting f appropriately, express f as a sine series.

(c) Plot three different partial sums of the sine series, clearly indicating the partial sums being plotted.

(d) Sketch a graph of the function to which the sine series converges for three periods.

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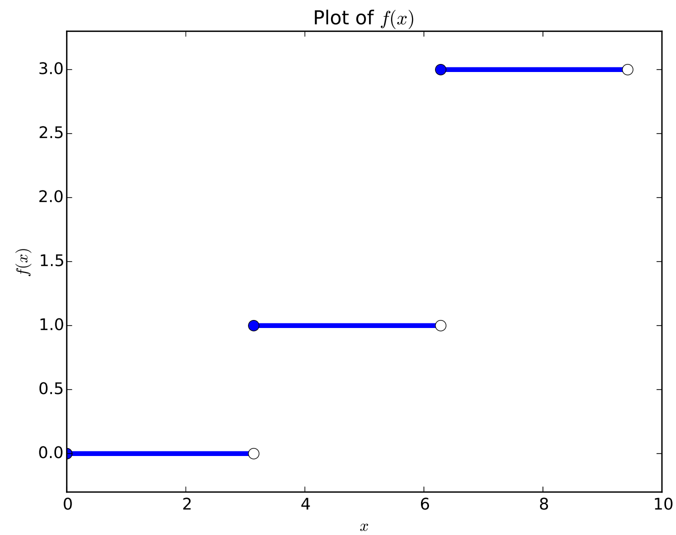
Solution 3.

(a) 

(b) To express $f(x)$ as a sine series, we create a new function $g(x)$ which is odd and periodic by reflecting $f(x)$ oddly accross the y-axis, and then defining $g(x+6\pi) = g(x)$ for all x. Since $g(x)$ is periodic, it has a Fourier series, and since $g(x)$ is odd, all of the cosine terms will be gone, leaving just the sine terms. We can calculate the associated coefficients by using the Euler-Fourier formula:

$$b_n = \frac{1}{3\pi}\int_{-3\pi}^{3\pi} g(x)\sin(n\pi x/(3\pi))dx.$$

Now since $g(x)$ is odd, the integrand is even, so we can simply integrate from 0 to 3π and multiply by 2 to get the value of b_n. Moreover, from 0 to 3π the function $g(x)$ agrees with $f(x)$, and therefore

$$b_n = \frac{2}{3\pi}\int_0^{3\pi} f(x)\sin(nx/3)dx.$$

Now in order to do this intergral, we need to break it up into the three separate intervals where $f(x)$ is individually defined:

$$b_n = \frac{2}{3\pi}\left(\int_0^{\pi} 0\sin(nx/3) + \int_{\pi}^{2\pi} 1\sin(nx/3)dx + \int_{2\pi}^{3\pi} 3\sin(nx/3)dx\right).$$

The integrals themselves are pretty easy. Evaluating them, we obtain:

$$b_n = \frac{2}{3\pi}\left(0 + \frac{-3}{n}(\cos(2n\pi/3) - \cos(n\pi/3))dx + \frac{-9}{n}(\cos(3n\pi/3) - \cos(2n\pi/3))\right).$$

Now we want to use the fact that

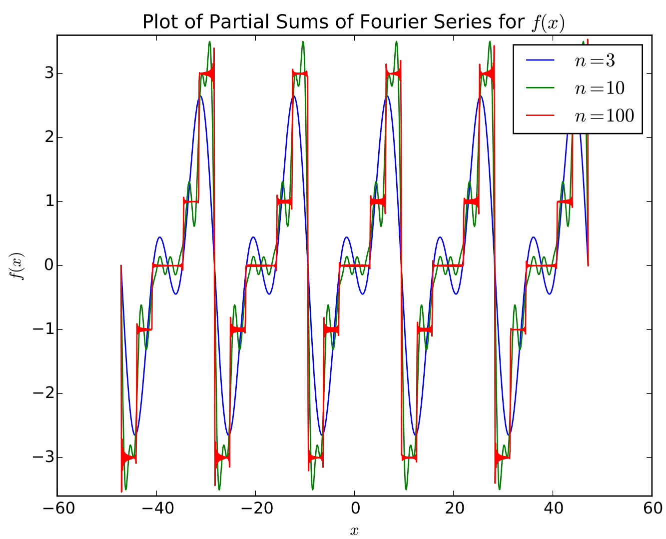
$$\cos(m\pi/3) = \begin{cases} 1/2, & m = \pm 1 + 6k \\ -1/2, & m = \pm 2 + 6k \\ 1, & m = 0 + 6k \\ -1, & m = 3 + 6k \end{cases}$$

Using this, the expression for b_n reduces to

$$b_n = \begin{cases} 5/(n\pi), & n = \pm 1 + 6k \\ -9/(n\pi), & n = \pm 2 + 6k \\ 0, & n = 0 + 6k \\ 8/(n\pi), & n = 3 + 6k \end{cases}$$

Using these values of b_n, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx/3).$$



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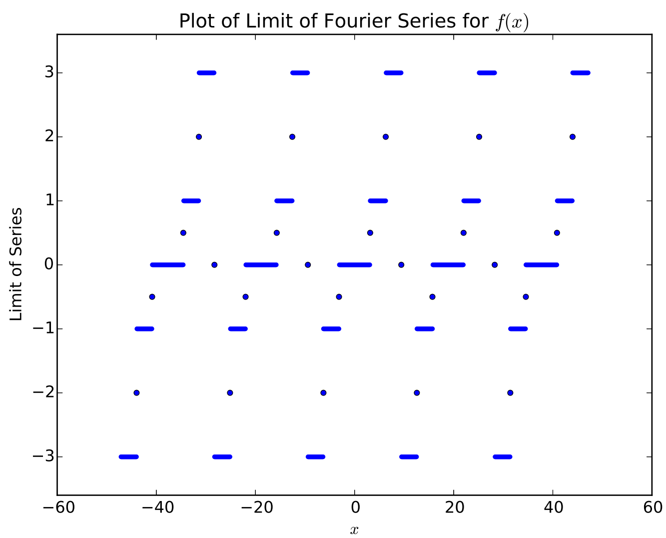

(d)

Problem 4 *Cosine Series*

Consider the function

$$f(x) = \begin{cases} x, 0 < x < \pi \\ 0, \pi < x < 2\pi \end{cases}$$

(a) Scketch a graph of $f(x)$

(b) By reflecting f appropriately, express f as a cosine series.

(c) Plot three different partial sums of the cosine series, clearly indicating the partial sums being plotted.

(d) Sketch a graph of the function to which the cosine series converges for three periods.

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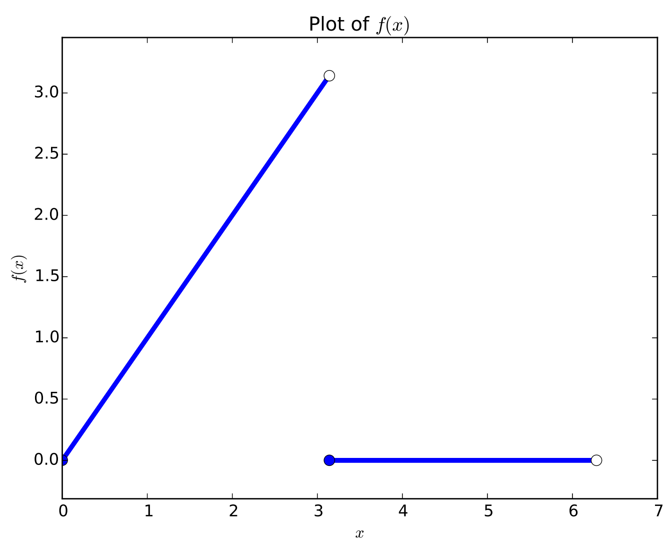
Solution 4.



(a)

(b) To express $f(x)$ as a cosine series, we create a new function $g(x)$ which is even and periodic by reflecting $f(x)$ evenly accross the y-axis, and then defining $g(x+4\pi) = g(x)$ for all x. Since $g(x)$ is periodic, it has a Fourier series, and since $g(x)$ is even, all of the sine terms will be gone, leaving just the cosine terms. We can calculate the associated coefficients by using the Euler-Fourier formula:

$$a_n = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} g(x) \cos(n\pi x/(2\pi)) dx.$$

Now since $g(x)$ is odd, the integrand is even, so we can simply integrate from 0 to 2π and multiply by 2 to get the value of a_n. Moreover, from 0 to 2π the function

$g(x)$ agrees with $f(x)$, and therefore

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx/2)dx.$$

Now in order to do this intergral, we need to break it up into the two separate intervals where $f(x)$ is individually defined:

$$a_n = \frac{1}{\pi}\left(\int_0^{\pi} x\cos(nx/2) + \int_{\pi}^{2\pi} 0\cos(nx/2)dx\right).$$

To evaluate this integral, we use integration by parts, obtaining:

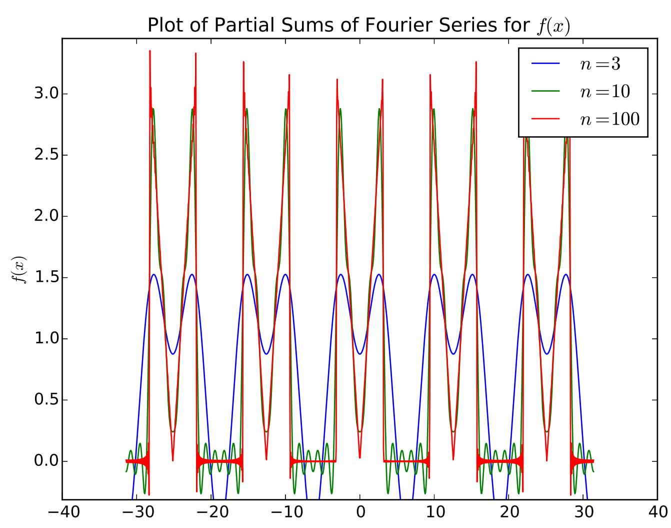
$$a_n = \frac{-2}{n}\cos(n\pi/2) = \begin{cases} ((-1)^{n/2}-1)4/(n^2\pi) & n \text{ even} \\ (-1)^{(n+1)/2}2/n - 4/(n^2\pi) & n \text{ odd} \end{cases}$$

This expression does not work however for $n = 0$ since in the calculation we divided by n. We must do this separately:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{1}{2}\pi.$$

Using these values of a_n, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x/3).$$



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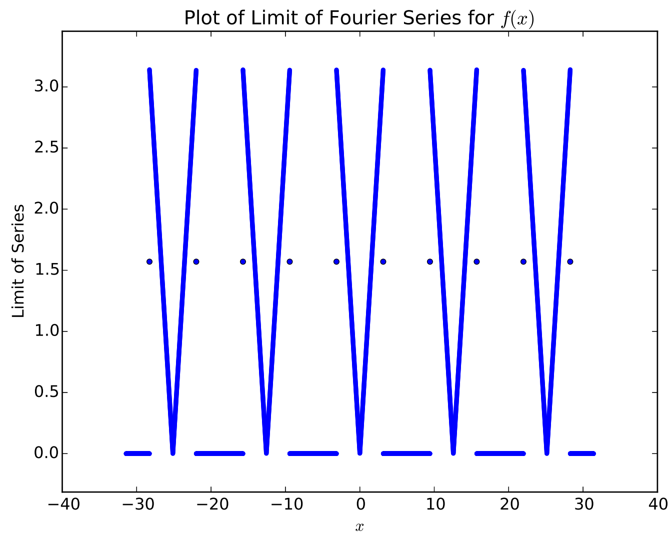

(d)

Problem 5 *Heat Equation 1*

Find the solution of the heat conduction problem

$$100u_{xx} = u_t, \quad 0 < x < 1, \ t > 0$$
$$u(0,t) = u(1,t) = 0, \ t > 0$$
$$u(x,0) = \sin(2\pi x) - \sin(5\pi x)$$

. . . .

Solution 5. We identify $\alpha^2 = 100$ and $L = 1$. Then we need to expand $u(x,0)$ as

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

However, if we look at the form of $u(x,0)$, this is immediately accomplished by taking $b_2 = 1, b_5 = -1$ and $b_n = 0$ otherwise. Therefore

$$u(x,t) = u_2(x,t) - u_5(x,t) = e^{-400\pi^2 t}\sin(2\pi x) - e^{-2500\pi^2 t}\sin(5\pi x).$$

Problem 6 *Heat Equation 2*

Find the solution of the heat conduction problem

$$u_{xx} = 4u_t, \quad 0 < x < 2, \ t > 0$$
$$u(0,t) = u(2,t) = 0, \ t > 0$$
$$u(x,0) = 2\sin(\pi x/2) - \sin(\pi x) + 4\sin(2\pi x)$$

. . . .

Solution 6. We identify $\alpha^2 = 4$ and $L = 2$. Then we need to expand $u(x,0)$ as

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2).$$

However, if we look at the form of $u(x,0)$, this is immediately accomplished by taking $b_1 = 2, b_2 = -1, b_4 = 4$ and $b_n = 0$ otherwise. Therefore

$$u(x,t) = u_1(x,t) - u_2(x,t) + 4u_4(x,t) = e^{-\pi^2 t}\sin(\pi x/2) - e^{-4\pi^2 t}\sin(\pi x) + 4e^{-16\pi^2 t}\sin(2\pi x).$$

Problem 7 *Schrödinger Equation*

In quantum mechanics, the position of a point particle in space is not certain – it's described by a probability distribution. The probability distribution of the position of the particle is $|\psi(x,t)|^2$, where $\psi(x,t)$ is the **wave function** of the particle. (Note: the wave function $\psi(x,t)$ can be complex-valued!!). The one-dimensional, time-dependent Schrödinger equation, describing the wave function $\psi(x,t)$ of a particle of mass m interacting with a potential $v(x)$ is given by

$$i\hbar\psi_t(x,t) = -\frac{\hbar^2}{2m}\psi_{xx}(x,t) + v(x)\psi(x,t)$$

where $\hbar$ is some universal constant. The potential $v(x)$ can be imagined as a function describing the particles interaction with whatever "stuff" is in the space surrounding the particle, eg. walls, external forces, etc.

(a) Use separation of variables to replace this partial differential equation with a pair of two ordinary differential equations

(b) If $v(x)$ is a potential corresponding to an "infinite square well":

$$v(x) = \begin{cases} 0, & -1 < x < 1 \\ \infty, & |x| \geq 1 \end{cases}$$

Then $\psi(x,t)$ must be zero whenever $|x| \geq 1$ and therefore $\psi(x,t)$ is the wave function of a particle trapped in a one-dimensional box! In other words, this potential describes a particle surrounded by impermeable walls. In this case, Schrödinger's equation reduces to

$$\begin{aligned} i\hbar\psi_t(x,t) &= -\frac{\hbar^2}{2m}\psi_{xx}(x,t), \quad -1 < x < 1,\ t > 0 \\ \psi(-1,t) &= \psi(1,t) = 0,\ t > 0 \end{aligned}$$

Suppose that initially the wave function is known to be

$$\psi(x,0) = \frac{3}{5}\sin(\pi x) + \frac{4}{5}\sin(3\pi x).$$

Determine $\psi(x,t)$ for all $t > 0$.

(c) Since $|\psi(x,t)|^2$ is the probability *distribution* of the particle's position at time t, the probability that the particle is somewhere in the box between ℓ_1 and ℓ_2 is given by

$$\mathbb{P}(\ell_1 \leq \text{pos} \leq \ell_2) = \int_{\ell_1}^{\ell_2} |\psi(x,t)|^2 dx.$$

Show that the probability $\mathbb{P}(-1 \leq \text{pos} \leq 1)$ that the particle is between -1 and 1 is always 1 (in other words, the particle is always in the box!).

(d) What is the probability $\mathbb{P}(-1 \leq \text{pos} \leq 0)$ that the particle is in the first half of the box at any given time?

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Solution 7.

(a) We assume $\psi(x,t) = F(x)G(t)$. Then inserting this into Schrödinger's equation, we obtain

$$i\hbar F(x)G'(t) = -\frac{\hbar^2}{2m}F''(x)G(t) + v(x)F(x)G(t).$$

Now if we divide out by a $G(t)$ and a $F(x)$ we find

$$i\hbar G'(t)/G(t) = -\frac{\hbar^2}{2m}F''(x)/F(x) + v(x).$$

The function on the left hand side is a function of t only. The function on the right hand side is a function of x only. Therefore the only way that the above equality can work is if both sides are equal to some constant E. Therefore

$$i\hbar G'(t)/G(t) = E, \quad -\frac{\hbar^2}{2m}F''(x)/F(x) + v(x) = E.$$

Simplifying, this gives us the system of two ordinary differential equations

$$i\hbar G'(t) = EG(t).$$

$$-\frac{\hbar^2}{2m}F''(x) + v(x)F(x) = EF(x).$$

The latter equation of these two equations is known as the **time-independent Schrödinger equation**.

(b) This is just like the heat equation, with $\alpha^2 = i\frac{\hbar}{2m}$ and $L = 1$. Thus given the initial condition, the solution that we are looking for is

$$\psi(x,t) = \frac{3}{5}e^{-i\frac{\hbar\pi^2}{2m}t}\sin(\pi x) + \frac{4}{5}e^{-i\frac{9\hbar\pi^2}{2m}t}\sin(3\pi x).$$

(c) Note that

$$\psi(x,t)^* = \frac{3}{5}e^{i\frac{\hbar\pi^2}{2m}t}\sin(\pi x) + \frac{4}{5}e^{i\frac{9\hbar\pi^2}{2m}t}\sin(3\pi x),$$

and therefore

$$|\psi(x,t)|^2 = \psi(x,t)\psi(x,t)^* = \frac{9}{25}\sin^2(\pi x)+\frac{16}{25}\sin^2(3\pi x)+\frac{12}{25}\left(e^{i\frac{8\hbar\pi^2}{2m}t} + e^{i\frac{-8\hbar\pi^2}{2m}t}\right)\sin(\pi x)\sin(3\pi x).$$

If we now integrate over the domain of the box (from -1 to 1), orthogonality tells us the integral of $\sin(\pi x)\sin(3\pi x)$ dies off! Therefore we obtain:

$$\int_{-1}^{1}|\psi(x,t)|^2dx = \frac{9}{25}\int_{-1}^{1}\sin^2(\pi x)dx + \frac{16}{25}\int_{-1}^{1}\sin^2(3\pi x)dx = \frac{9}{25}+\frac{16}{25} = 1.$$

This shows that the probability that the particle is in the box at any time t is 1 – e.g. it is a certainty.

(d) We can use the work from above to write

$$\int_{-1}^{1}|\psi(x,t)^2dx = \frac{9}{25}\int_{-1}^{0}\sin^2(\pi x)dx+\frac{16}{25}\int_{-1}^{0}\sin^2(3\pi x)dx+\frac{12}{25}\left(e^{i\frac{8\hbar\pi^2}{2m}t} + e^{i\frac{-8\hbar\pi^2}{2m}t}\right)\int_{-1}^{0}\sin(\pi x)\sin(3\pi$$

However, since we're not integrating over the full period, we cannot appeal to orthogonality to say that the cross-term dies anymore. However, direct calculation shows that it does indeed die anyway. The sum of the first two integrals is easily calculated to be 1/2. Therefore the probability that the particle is in the first half of the box at any time t is exactly 1/2. In other words – at any time the particle is equally likely to be in either side of the box.