MATH 309: Homework #5

Due on: November 30, 2015

Problem 1 *Insulated Heat Equation Problem*

Consider a uniform rod of length L with an initial temperature given by $u(x, 0) = \sin(\pi x/L)$ with $0 \leq x \leq L$. Assume that both ends of the bar are insulated (this is a homogeneous von Neumann boundary condition for $t > 0$).

(a) Find the temperature $u(x, t)$. (Note: the initial condition $u(x, 0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Let $\alpha^2 = 1$ and $L = 40$. Plot u vs. x for several values of t.

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Problem 2 *Another Insulated Heat Equation Problem*

Consider a bar of length 40 cm whose initial temperatore is given by $u(x, 0) = x(60 - x)/30$. Suppose that $\alpha^2 = 1/4$ cm^2/s and that both ends of the bar are insulated.

(a) Find the temperature $u(x, t)$. (Note: the initial condition $u(x, 0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Plot u vs. x for several values of t.

(d) Determine how much time must elapse before the temperature at $x = 40$ comes within 1 degrees C of its steady state value.

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Problem 3 *Nonhomogeneous Boundary Conditions*

Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25 degrees C. Suppose that at time $t = 0$, the end $x = 0$ is cooled to 0 degrees C while the other end $x = 20$ is heated to 60 degrees C, and both are thereafter maintained at those temperatures.

(a) Find the temperature $u(x, t)$. (Note: the initial condition $u(x, 0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Plot u vs. x for several values of t.

(d) Determine how much time must elapse before the temperature at $x = 5$ comes within 1 degrees C of its steady state value.

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Problem 4 *The Heat Equation in Two Dimensions*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Assume that u is of the form $u(x, y, t) = F(x)G(y)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\begin{cases} T'(t) + \lambda T = 0 \\ F''(x) + \frac{\lambda-\mu}{\alpha^2}F(x) = 0 \\ G''(y) + \frac{\mu}{\alpha^2}G(y) = 0 \end{cases}$$

for some constants λ and μ.

(b) Assume that $u(x, y, t) = F(x)G(y)T(t)$ satisfies the heat equation above in the rectangular region $[0, L] \times [0, M]$ and also satisfies the Dirichlet boundary conditions

$$u(0, y, t) = 0, u(L, y, t) = 0, u(x, 0, t) = 0, u(x, M, t) = 0.$$

Find all possible functions $u(x, y, t)$ satisfying the above conditions. [Hint: they should be indexed by pairs of positive integers (m, n)]

(c) Use (b) to find a solution to the two dimensional heat equation with Dirichlet boundary conditions

$$u_t - (u_{xx} + u_{yy}) = 0,$$

$$u(0, y, t) = 0, u(1, y, t) = 0, u(x, 0, t) = 0, u(x, 1, t) = 0,$$

with the initial condition that

$$u(x, y, 0) = \sin(3\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(4\pi y).$$

Create a surface plots of your solution for several values of t.

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Problem 5 *The Heat Equation in Polar Coordinates*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Show that using polar coordinates, (r, θ), the heat equation becomes

$$u_t - \alpha^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = 0.$$

(b) Assume that u is of the form $u(r, \theta, t) = R(r)S(\theta)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\begin{cases} T'(t) + \lambda T = 0 \\ r^2R''(r) + rR'(r) + \frac{1}{\alpha^2}(r^2\lambda - \mu)R = 0 \\ S''(\theta) + \frac{\mu}{\alpha^2}S(\theta) = 0 \end{cases}$$

for some constants λ and μ.

(c) Explain why $\mu = n^2\alpha^2$ for some integer n. [Hint: remember that θ is the angle counter-clockwise from the x-axis].

(d) Find the general solution to the above system of equations in the case that $\lambda = 0$ and $\mu = \alpha^2$. [Hint: to solve for $R(r)$, propose a solution of the form $R(r) = r^b$]

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