MATH 309: Homework #5

Due on: November 30, 2015

Problem 1 *Insulated Heat Equation Problem*

Consider a uniform rod of length L with an initial temperature given by $u(x, 0) = \sin(\pi x/L)$ with $0 \le x \le L$. Assume that both ends of the bar are insulated (this is a homogeneous von Neumann boundary condition for $t > 0$).

(a) Find the temperature $u(x, t)$. (Note: the initial condition $u(x, 0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Let $\alpha^2 = 1$ and $L = 40$. Plot u vs. x for several values of t.

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Solution 1.

(a) We need to determine the temperature initially in terms of a cosine series. This means reflecting $\sin(\pi x/L)$ *evenly* and then extending periodically. In other words, we're really looking for the cosine series of $|\sin(\pi x/L))|$. Using Euler-Fourier, we obtain

$$a_n = \frac{1}{L}\int_{-L}^{L} |\sin(\pi x/L)| \cos(n\pi x/L)dx = \frac{2}{L}\int_0^L \sin(\pi x/L)\cos(n\pi x/L)dx.$$

Now in order to complete the last integral on the right, we can adopt several strategies. The most obvious thing is to integrate by parts twice, and then compare sides – however, that is a lot of work. A shorter strategy is to use the addition angle formulas for sine to write:

$$\sin(\pi x/L)\cos(n\pi x/L) = \frac{1}{2}(\sin((1+n)\pi x/L) + \sin((1-n)\pi x/L)).$$

With this in mind, the above integral becomes

$$a_n = \frac{1}{L}\int_0^L (\sin((1+n)\pi x/L) + \sin((1-n)\pi x/L))dx = \frac{2}{\pi}\left(\frac{1+(-1)^n}{1-n^2}\right).$$

However, notice that in our derivation of this formula, we divided by $1 - n$, and therefore the expression we obtained for a_n does not apply when $n = 1$. We must treat this case separately! We calculate using the double angle formula

$$a_1 = \frac{2}{L}\int_0^L \sin(\pi x/L)\cos(\pi x/L)dx = \frac{1}{L}\int_0^L \sin(2\pi x/L)dx = -\frac{1}{2\pi}\cos(2\pi x/L)|_0^1 = 0.$$

We conclude that

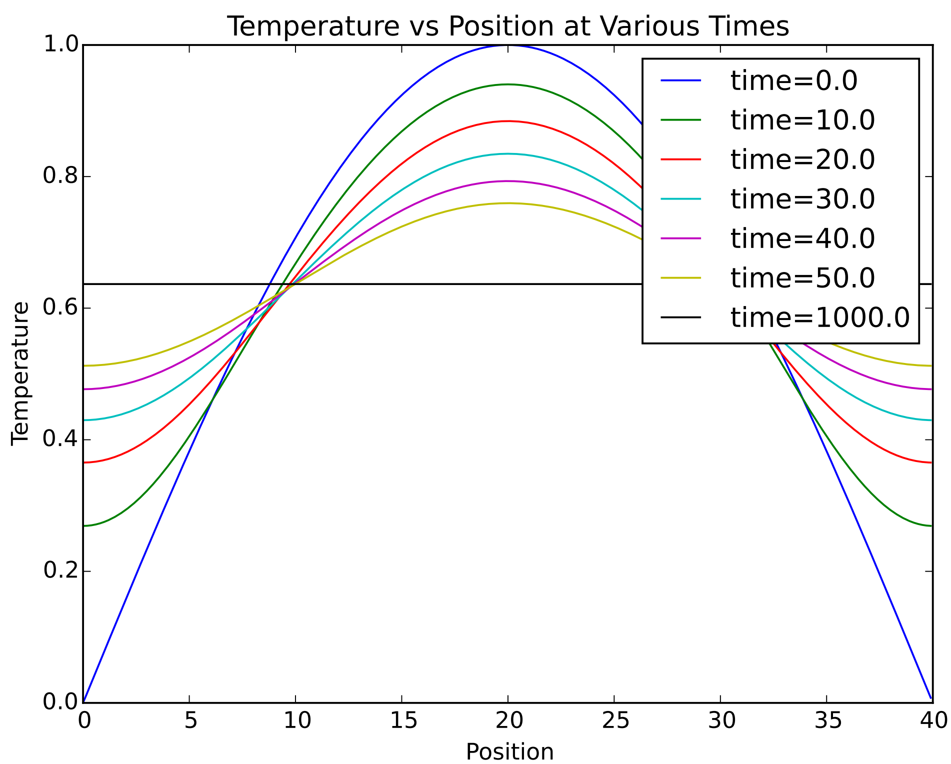
$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi}\left(\frac{1+(-1)^n}{1-n^2}\right)\cos(n\pi x/L).$$

This tells us that

$$u(x,t) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi}\left(\frac{1+(-1)^n}{1-n^2}\right) e^{-n^2\pi^2\alpha^2 t/L^2}\cos(n\pi x/L).$$

(b) As $t \to \infty$, the exponential terms die off, leaving only $a_0/2$. Therefore the steady state temperature is $2/\pi$.

(c) Plot at several times is included in the figure below.



Problem 2 *Another Insulated Heat Equation Problem*

Consider a bar of length 40 cm whose initial temperatore is given by $u(x, 0) = x(60 - x)/30$. Suppose that $\alpha^2 = 1/4$ cm^2/s and that both ends of the bar are insulated.

(a) Find the temperature $u(x, t)$. (Note: the initial condition $u(x, 0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Plot u vs. x for several values of t.

(d) Determine how much time must elapse before the temperature at $x = 40$ comes within 1 degrees C of its steady state value.

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Solution 2.

(a) Again, we must extend $u(x, 0)$ evenly and periodically in order to pick up its cosine series. Then by the Euler-Fourier equation we have

$$a_n = \frac{2}{40}\int_0^{40} \frac{x(60-x)}{30}\cos(n\pi x/40)dx.$$

We can obtain the value explicitly by using integration by parts twice to get the $a_n's$. (There are, of course, more clever ways to do things, but this works fine). Doing so, we obtain

$$a_n = \frac{160}{3}\frac{(-1)^{n+1} - 3}{n^2\pi^2},$$

which works except for $n = 0$, for which we obtain $a_0 = 400/9$. Therefore we see

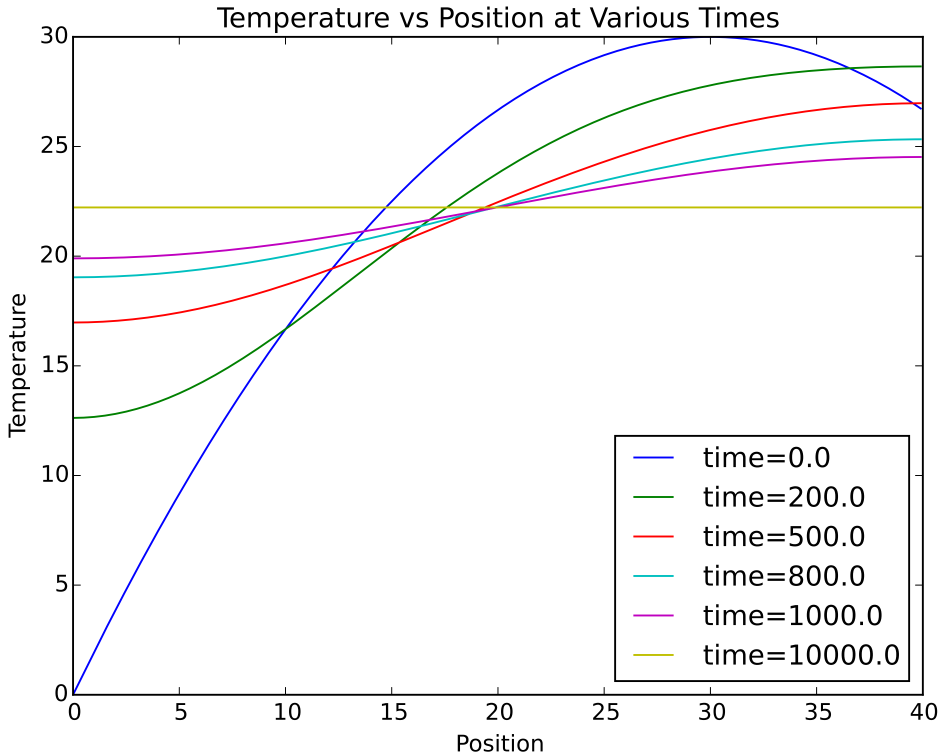
$$u(x, 0) = 200/9 + \sum_{n=1}^{\infty} \frac{160}{3}\frac{(-1)^{n+1} - 3}{n^2\pi^2}\cos(n\pi x/40).$$

We conclude that

$$u(x, t) = 200/9 + \sum_{n=1}^{\infty} \frac{160}{3}\frac{(-1)^{n+1} - 3}{n^2\pi^2}e^{-n^2\pi^2 t/6400}\cos(n\pi x/40).$$

(b) Again, the exponential terms die off, so the steady state temperature is 200/9.

(c) Plot at several times is included in the figure below.



Problem 3 *Nonhomogeneous Boundary Conditions*

Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25 degrees C. Suppose that at time $t = 0$, the end $x = 0$ is cooled to 0 degrees C while the other end $x = 20$ is heated to 60 degrees C, and both are thereafter maintained at those temperatures.

(a) Find the temperature $u(x, t)$. (Note: the initial condition $u(x, 0)$ does not satisfy the boundary conditions, which is fine since we are only asking the boundary conditions to be satisfied for $t > 0$)

(b) What is the steady state temperature as $t \to \infty$?

(c) Plot u vs. x for several values of t.

(d) Determine how much time must elapse before the temperature at $x = 5$ comes within 1 degrees C of its steady state value.

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Solution 3.

(a) We have Dirichlet boundary conditions, so this means that we should be thinking about a sine expansion. The first thing we should do is obtain the steady state solution $u_{\text{steady}}(x)$. Note that it is **not** time dependent (since it's steady state!!). Since it satisfies the heat equation, we know that $u''_{\text{steady}}(x) = 0$, and therefore $u_{\text{steady}}(x) = ax + b$ for some constants a and b. Now since $u_{\text{steady}}(0) = 0$ and $u_{\text{steady}}(20) = 60$, we can work out a and b, obtaining $(a = 3, b = 0)$:

$$u_{\text{steady}}(x) = 3x.$$

Next, we should solve for the transient solution, which has homogeneous Dirichlet boundary conditions and initially satisfies $u_{\text{trans}}(x, 0) = 25 - u_{\text{steady}}(x) = 25 - 3x$. The sine expansion of this is given by

$$u_{\text{trans}}(x, 0) = \sum_{n=1}^{\infty} \frac{10}{n\pi}(5 + 7(-1)^n) \sin(n\pi x/20).$$

Then since $\alpha^2 = 0.86$ for aluminum, we find that the transient solution is:

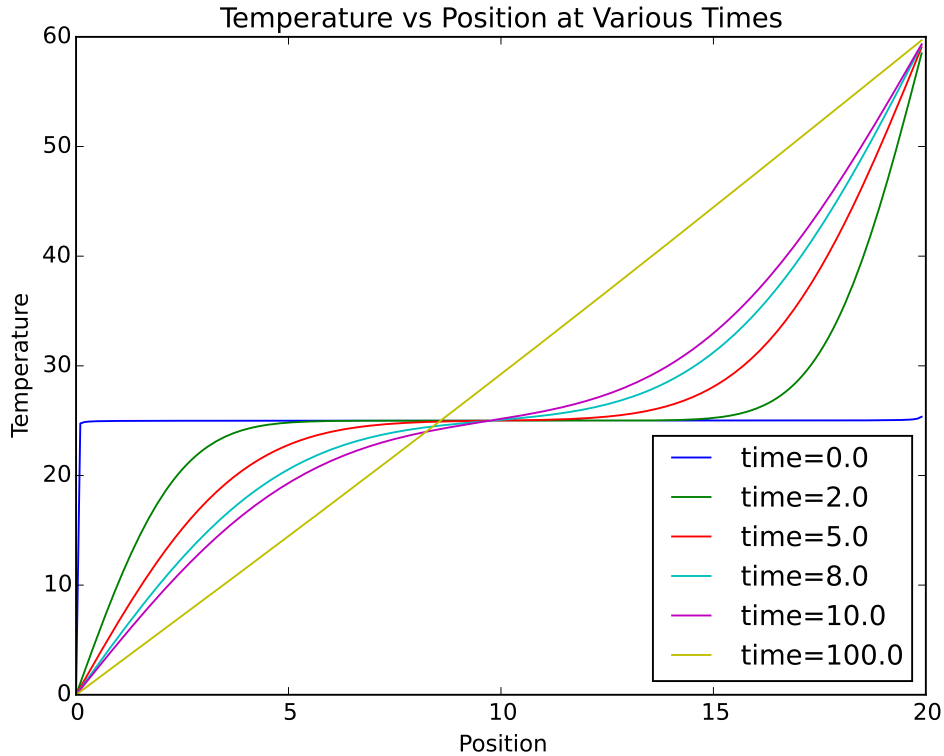
$$u_{\text{trans}}(x, t) = \sum_{n=1}^{\infty} \frac{10}{n\pi}(5 + 7(-1)^n)e^{-n^2\pi^2(0.86)t/400} \sin(n\pi x/20).$$

To get the solution to the inhomogeneous heat equation problem above, we need to add this transient solution to the steady state solution $u_{\text{steady}}(x)$. Therefore

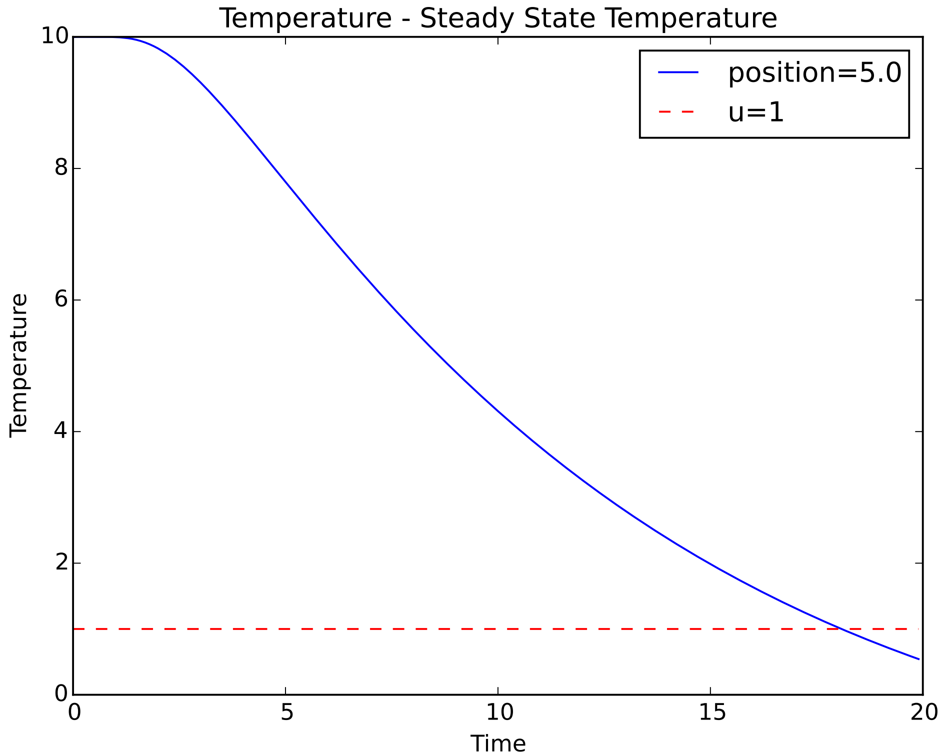
$$u = u_{\text{trans}}(x, t) + u_{\text{steady}}(x, t) = 3x + \sum_{n=1}^{\infty} \frac{10}{n\pi}(5 + 7(-1)^n)e^{-n^2\pi^2(0.86)t/400} \sin(n\pi x/20).$$

(b) As $t \to \infty$, the transient solution dies off, leaving behind only the steady state solution $u_{\text{steady}} = 3x$.

(c) Plot at several times is included in the figure below.



(d) We are really asking how much time elapses until $|u_{\text{trans}}(5,t)| \le 1$. We can determine this by plotting $u_{\text{trans}}(5,t)$ and determining what time it drops below 1 degree Celsius. A graph of the transient temperature is included below:



From the graph, we approximate $t = 18.18$ is about when $u(5,t)$ is within its steady state value.

Problem 4 *The Heat Equation in Two Dimensions*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Assume that u is of the form $u(x,y,t) = F(x)G(y)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\begin{cases} T'(t) + \lambda T = 0 \\ F''(x) + \frac{\lambda-\mu}{\alpha^2}F(x) = 0 \\ G''(y) + \frac{\mu}{\alpha^2}G(y) = 0 \end{cases}$$

for some constants λ and μ.

(b) Assume that $u(x,y,t) = F(x)G(y)T(t)$ satisfies the heat equation above in the rectangular region $[0,L] \times [0,M]$ and also satisfies the Dirichlet boundary conditions

$$u(0,y,t) = 0, u(L,y,t) = 0, u(x,0,t) = 0, u(x,M,t) = 0.$$

Find all possible functions $u(x,y,t)$ satisfying the above conditions. [Hint: they should be indexed by pairs of positive integers (m,n)]

(c) Use (b) to find a solution to the two dimensional heat equation with Dirichlet boundary conditions

$$u_t - (u_{xx} + u_{yy}) = 0,$$
$$u(0, y, t) = 0, u(1, y, t) = 0, u(x, 0, t) = 0, u(x, 1, t) = 0,$$

with the initial condition that

$$u(x, y, 0) = \sin(3\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(4\pi y).$$

Create a surface plots of your solution for several values of t.

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Solution 4.

(a) Pluggin $u(x, y, t) = F(x)G(y)T(t)$ into the two dimensional heat equation, we obtain

$$F(x)G(y)T'(t) = \alpha^2(F''(x)G(y)T(t) + F(x)G''(y)T(t)).$$

Then dividing on both sides by $F(x)G(y)T(t)$, we obtain:

$$T'(t)/T(t) = \alpha^2(F''(x)/F(x) + G''(y)/G(y)).$$

The expression on the left hand side is a function of t only, while the expression on the right hand side is independent of t. Therefore both must be equal to a constant $-\lambda$:

$$T'(t)/T(t) = \alpha^2(F''(x)/F(x) + G''(y)/G(y)) = -\lambda.$$

Simplifying this, we get

$$T'(t)/T(t) = -\lambda$$
$$\alpha^2 F''(x)/F(x) + \alpha^2 G''(y)/G(y) = -\lambda.$$

Therefore

$$\alpha^2 F''(x)/F(x) = -\alpha^2 G''(y)/G(y) - \lambda.$$

Again the expression on the left hand side is a function of x only, and on the right we have a function of y only, and therefore both are equal to a constant μ:

$$\alpha^2 F''(x)/F(x) + \lambda = -\alpha^2 G''(y)/G(y) = \mu.$$

It follows that

$$\alpha^2 F''(x)/F(x) = \mu - \lambda$$
$$-\alpha^2 G''(y)/G(y) = -\mu.$$

Simplyfying things further we obtain the system of three ordinary differential equations listed in (a) above.

(b) The Dirichlet boundary conditions result in boundary conditions on our various ODE's. In particular

$$F''(x) + \frac{\lambda-\mu}{\alpha^2}F(x) = 0,\ F(0) = 0,\ F(L) = 0$$

and also

$$G''(y) + \frac{\mu}{\alpha^2}G(y) = 0,\ G(0) = 0,\ G(M) = 0.$$

Then from our experience with boundary value problems, to get a nontrivial solution this says that $\mu = n^2\pi^2\alpha^2/M^2$ and that $\lambda - \mu = m^2\pi^2\alpha^2/L^2$ for some integers m and n, and in this case the solutions we obtain are

$$F(x) = A\sin(m\pi x/L), \quad G(x) = B\sin(n\pi y/M)$$

for some constants A and B. Then $\lambda = m^2\pi^2\alpha^2/L^2 + n^2\pi^2\alpha^2/M^2$, and therefore

$$T = C\exp\left(\frac{m^2\pi^2\alpha^2}{L^2}t + \frac{n^2\pi^2\alpha^2}{M^2}t\right).$$

Thus we obtain the solution

$$u(x, y, t) = F(x)G(y)T(t) = ABC\exp\left(\frac{m^2\pi^2\alpha^2}{L^2}t + \frac{n^2\pi^2\alpha^2}{M^2}t\right)\sin(m\pi x/L)\sin(n\pi x/M).$$

This motivates us to set

$$u_{mn}(x, y, t) = \exp\left(\frac{m^2\pi^2\alpha^2}{L^2}t + \frac{n^2\pi^2\alpha^2}{M^2}t\right)\sin(m\pi x/L)\sin(n\pi x/M).$$

Then by the superposition principle, any solution of the form

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}u_{mn}(x, y, t)$$

is a solution, for constants a_{mn}. In fact, all solutions may be written this way!

(c) From part (b), we know to try to write

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}u_{mn}(x, t)$$

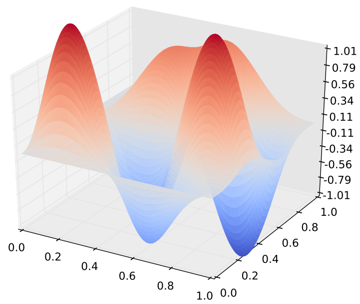
for some constants a_{mn}, where here $\alpha^2 = 1, L = 1, M = 1$. we must choose the constants so that

$$u(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}u_{mn}(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin(m\pi x)\sin(n\pi x).$$

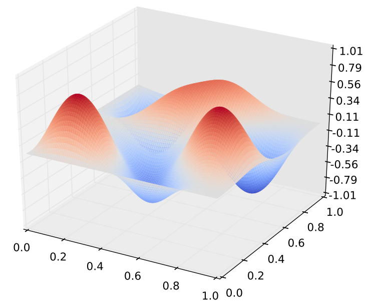
is equal to our initial condition. Looking at initial condition, this is easy! Just choose $a_{32} = 1, a_{24} = 1$ and $a_{mn} = 0$ otherwise. Thus:

$$u(x, y, t) = u_{32}(x, y, t) + u_{24}(x, y, t) = e^{-13\pi^2 t}\sin(3\pi x)\sin(2\pi y) + e^{-20\pi^2 t}\sin(2\pi x)\sin(4\pi y).$$

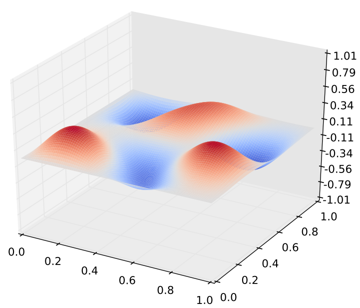
Plots at various times are included below:



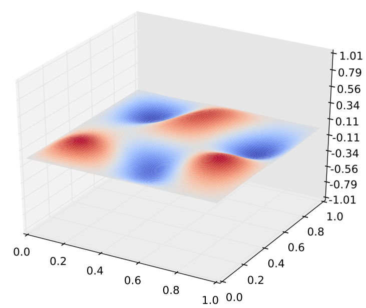
$t = 0.000$



$t = 0.005$



$t = 0.010$



$t = 0.015$

Problem 5 *The Heat Equation in Polar Coordinates*

We consider the two dimensional heat equation

$$u_t - \alpha^2(u_{xx} + u_{yy}) = 0.$$

(a) Show that using polar coordinates, (r, θ), the heat equation becomes

$$u_t - \alpha^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) = 0.$$

(b) Assume that u is of the form $u(r, \theta, t) = R(r)S(\theta)T(t)$, and show that the heat equation reduces to the system of three ordinary differential equations

$$\begin{cases} T'(t) + \lambda T = 0 \\ r^2 R''(r) + rR'(r) + \frac{1}{\alpha^2}(r^2\lambda - \mu)R = 0 \\ S''(\theta) + \frac{\mu}{\alpha^2} S(\theta) = 0 \end{cases}$$

for some constants λ and μ.

(c) Explain why $\mu = n^2\alpha^2$ for some integer n. [Hint: remember that θ is the angle counter-clockwise from the x-axis].

(d) Find the general solution to the above system of equations in the case that $\lambda = 0$ and $\mu = \alpha^2$. [Hint: to solve for $R(r)$, propose a solution of the form $R(r) = r^b$]

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Solution 5.

(a) Note that $r^2 = x^2 + y^2$ and $\tan(\theta) = y/x$ and therefore

$$r_x = \cos(\theta), \quad r_y = \sin(\theta),$$

as well as

$$\theta_x = \frac{-\sin(\theta)}{r}, \quad \theta_y = \frac{\cos(\theta)}{r},$$

Then from the chain rule we have

$$\frac{\partial}{\partial x} = r_x \frac{\partial}{\partial r} + \theta_x \frac{\partial}{\partial \theta} = \cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = r_y \frac{\partial}{\partial r} + \theta_y \frac{\partial}{\partial \theta} = \sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}$$

Then we have that

$$\begin{aligned}\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)^2 &= \left[\cos(\theta)\frac{\partial}{\partial r} + \frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]^2 \\ &= \left[\cos(\theta)\frac{\partial}{\partial r}\right]^2 + \left[\cos(\theta)\frac{\partial}{\partial r}\right]\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right] \\ &\quad + \left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]\left[\cos(\theta)\frac{\partial}{\partial r}\right] + \left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]^2\end{aligned}$$

and also

$$\left[\cos(\theta)\frac{\partial}{\partial r}\right]^2 = \cos^2(\theta)\frac{\partial^2}{\partial r^2}.$$

$$\left[\cos(\theta)\frac{\partial}{\partial r}\right]\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right] = \frac{-1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial r\partial \theta} + \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}$$

$$\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]\left[\cos(\theta)\frac{\partial}{\partial r}\right] = \frac{-1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial r\partial \theta} + \frac{1}{r}\sin^2(\theta)\frac{\partial}{\partial r}.$$

$$\left[\frac{-1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right]^2 = \frac{1}{r^2}\sin^2(\theta)\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}.$$

Similarly,

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial y}\right)^2 = \left[\sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]^2$$
$$= \left[\sin(\theta)\frac{\partial}{\partial r}\right]^2 + \left[\sin(\theta)\frac{\partial}{\partial r}\right]\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]$$
$$+ \left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]\left[\sin(\theta)\frac{\partial}{\partial r}\right] + \left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]^2$$

and also

$$\left[\sin(\theta)\frac{\partial}{\partial r}\right]^2 = \sin^2(\theta)\frac{\partial^2}{\partial r^2}.$$

$$\left[\sin(\theta)\frac{\partial}{\partial r}\right]\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right] = \frac{1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}.$$

$$\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]\left[\sin(\theta)\frac{\partial}{\partial r}\right] = \frac{1}{r}\sin(\theta)\cos(\theta) + \frac{1}{r}\cos^2(\theta)\frac{\partial}{\partial r}.$$

$$\left[\frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right]^2 = \frac{1}{r^2}\cos^2(\theta)\frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}.$$

Adding all of this together, we obtain:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Applying this to the function u, we get

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Suubstiuting this into the heat equation leads to

$$u_t - \alpha^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = 0.$$

(b) Putting $u(r, \theta, t) = R(r)S(\theta)T(t)$ into the above expression, we obtain:

$$R(r)S(\theta)T'(t) = \alpha^2(R''(r)S(\theta)T(t) + \frac{1}{r}R'(r)S(\theta)T(t) + \frac{1}{r^2}R(r)S''(\theta)T(t)).$$

Dividing everything by $R(r)S(s)T(t)$, we obtain

$$T'(t)/T(t) = \alpha^2(R''(r)/R(r) + \frac{1}{r}R'(r)/R(r) + \frac{1}{r^2}S''(\theta)/S(\theta)).$$

The expression on the left hand side is a function of t only, while the right hand side is a function of r and θ only, and therefore both are equal to an arbitrary constant $-\lambda$. This leads to

$$T'(t)/T(t) = -\lambda$$

and also

$$\alpha^2(R''(r)/R(r) + \frac{1}{r}R'(r)/R(r) + \frac{1}{r^2}S''(\theta)/S(\theta)) = -\lambda.$$

Then after an algebraic manipulation,

$$\alpha^2 r^2 R''(r)/R(r) - \alpha^2 r R'(r)/R(r) - r^2\lambda = S''(\theta)/S(\theta).$$

Therefore again we have that both are equal to a constant $-\mu$:

$$\alpha^2 r^2 R''(r)/R(r) - \alpha^2 r R'(r)/R(r) - r^2\lambda = -\mu$$

and also

$$S''(\theta)/S(\theta) = -\mu.$$

After some simplification, this reduces to the expression in (b) above.

(c) Since (r, θ) and $(r, \theta + 2\pi)$ both refer to the same coordinate in polar coordinates, the function S should satisfy $S(\theta) = S(\theta + 2\pi)$. Moreover, we know that S satisfies $S''(\theta) + (\mu/\alpha^2)S(\theta) = 0$. Depending on the value of μ/α^2, the solutions to this are either exponentials, polynomials or trig. functions. Since we require S to be periodic, we want them to be trig functions, and therefore we need μ/α^2 to be positive. In this case, the general solution to the differential equation is

$$S(\theta) = A\cos(\sqrt{\mu}\theta/\alpha) + B\sin(\sqrt{\mu}\theta/\alpha).$$

Now we need S to be 2π periodic, and therefore we need $\sqrt{\mu}/\alpha = n$ for some integer n. Therefore $\mu = n^2\alpha^2$.

(d) Since $\lambda = 0$, the solution for T is $T = E$ for some constant E. Moreover, the general solution for S is

$$S(\theta) = A\cos(\theta) + B\sin(\theta)$$

Finally, we propose a solution $R(r) = r^b$ to the equation $r^2R''(r) + rR'(r) - R(r) = 0$, and therefore $b(b-1)r^{b-2}r^b + br^b - r^b = 0$. This leads to $b^2 - 1 = 0$, and therefore $b = \pm 1$. Thus we obtain two solutions: r and r^{-1}. The general solution is therefore

$$R(r) = Cr + Dr^{-1}$$

. Putting this all together, we obtain

$$u(r, \theta, t) = R(r)S(\theta)T(t) = (Cr + Dr^{-1})(A\cos(\theta) + B\sin(\theta))E.$$

Since A, B, C, D, E are all arbitrary constants, we can rewrite this as:

$$u(r, \theta, t) = C_1 r\cos(\theta) + C_2 r\sin(\theta) + C_3 r^{-1}\cos(\theta) + C_4 r^{-1}\sin(\theta).$$