MATH 309: Homework #6

Due on: December 7, 2015

Problem 1 *The Wave Equation I*

Consider an elastic string of length $L = 10$ whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x,0) = f(x)$, and the material properties of the string make $u(x,t)$ satisfy the wave equation $u_{tt} - c^2 u_{xx}$ with $c = 1$. For each of the values of $f(x)$ below, determine

(i) Determine the solution $u(x,t)$ in terms of an infinite linear combination of the fundamental set of solutions $u_n(x,t) = \sin(n\pi x/L)\cos(cn\pi t/L)$

(ii) Plot $u(x,t)$ vs. x for $t = 0, 4, 8, 12, 16$

(iii) Describe the motion of the string in a few sentences.

(a)

$$f(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2(L-x), & L/2 < x \le L \end{cases}$$

(b)

$$f(x) = 8x(L-x)^2/L^3.$$

(c)

$$f(x) = \begin{cases} 1, & |x - L/2| < 1 \\ 0, & |x - L/2| \ge 1 \end{cases}$$

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Problem 2 *The Wave Equation II*

Consider an elastic string of length $L = 10$ whose ends are held fixed. The string is set in motion from its equilibrium position with initial velocity given by $u_t(x,0) = g(x)$, and the material properties of the string make $u(x,t)$ satisfy the wave equation $u_{tt} - c^2 u_{xx}$ with $c = 1$. For each of the values of $g(x)$ below, determine

(i) Determine the solution $u(x,t)$ for $0 \le x \le L$ and $t > 0$ in terms of an infinite linear combination of the fundamental set of solutions $u_n(x,t) = \sin(n\pi x/L)\sin(cn\pi t/L)$

(ii) Plot $u(x,t)$ vs. x for $t = 0, 4, 8, 12, 16$

(iii) Describe the motion of the string in a few sentences.

(a)

$$g(x) = \begin{cases} 2x/L, & 0 \le x \le L/2 \\ 2(L-x), & L/2 < x \le L \end{cases}$$

(b)

$$g(x) = 8x(L-x)^2/L^3.$$

(c)

$$g(x) = \begin{cases} 1, & |x - L/2| < 1 \\ 0, & |x - L/2| \ge 1 \end{cases}$$

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Problem 3 Some Physics Flavor

A steel wire 5 ft in length is stretched by a tensile force of 50 lb. The wire has a weight per unit lenght of 0.026 lb/ft.

(a) Find the velocity of propagation of transverse waves in the wire.

(b) Find the natrual frequencies of vibration.

(c) If the tension in the wire is increased, how are the natural frequencies changed? Are the natural modes also changed?

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Problem 4 D'Alembert's Method

Use D'Alembert's Method to find a solution to the wave equation

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \ t > 0$$

satisfying $u(0) = 0$ and $u(1) = 0$, with the property that $u(x,0) = \sin^3(\pi x)$. Use this solution to create a surface plot of $u(x,t)$ for $0 \le x \le 1$ and $0 \le t \le 4$.

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Problem 5 Wave Equation with von Neummann Boundary Conditions

Use separation of variables to find a solution to the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

with the homogeneous von Neumman boundary conditions

$$u_x(0,t) = 0, \quad u_x(L,t) = 0,$$

and satisfying the initial condition

$$u(x,0) = \cos(n\pi x/L), u_t(x,0) = 0,$$

where here n is a nonnegative integer.

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