Math 309 Lecture 1

Welcome to Math 309!

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Today!

Plan for today:

- What is this class about?
- Review of Matrices

Next time:

- Systems of Linear Algebraic Equations
- Linear Independence
- Eigenvectors and Eigenvalues

Outline

Overview

In this class, we will study *linear* equations:

- Linear systems of algebraic equations
- Linear systems of differential equations
- Nonlinear equations which can be approximated linearly
- Linear partial differential equations

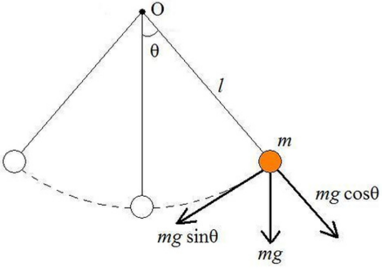
Question

Why should we care about linear equations?

Because they show up **naturally** all over the place!

Example Diff. Eqn: Motion of a Rigid Pendulum

Figure : A physics-type picture you've probably seen before



- Newton's second law: $\tau = I\frac{d^2\theta}{dt^2}$
- Torque: $\tau = mgl\sin\theta \approx mgl\theta$ (assuming small θ)
- Moment of inertia: $I = ml^2$
- We get a linear differential equation!

$$\frac{d^2\theta}{dt^2} = \frac{mg}{I}\theta$$

Example Diff. Eqn: Compound interest

Figure : A traditional celebration of compound interest as demonstrated by the notable entrepreneur Scrooge Mc. Duck

- For continuously compounded interest

$$\frac{dS}{dt} = rS$$

- S is invested capital
- r is interest rate
- This is a linear differential equation!
- Solution is $S(t) = S_0 e^{rt}$ (How do we get this?)

Example Diff. Eqn: Falling with air drag

Figure : Differential equations can help us answer important safety questions about the Red Bull Stratos Jump

- Newton's second law: $F = ma$
- Using a linear drag model

$$m\frac{d^2y}{dt^t} = -mg + k\frac{dy}{dt}$$

- y is your height
- g is gravitational acceleration
- k is a drag coefficient
- How can we solve this

Example Diff. Eqn: Fluid flow in one dimension

Figure : A fluid flow is as cool as it is complicated! Below is an example of what are called Von Karman vortices. Caveat: this flow is nonlinear

- Goal: find velocity of the fluid $u = u(x, t)$
- x, t, p, ρ are position, time, pressure, and density

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{d^2 p}{dx^2}$$

- It's a *partial differential equation* because it has partial derivatives
- It's nonlinear – we can

Our Main Tool

Question

What is our main tool for solving linear equations?

- That we can construct new solutions from old ones!
- We do this by taking *linear combinations*.
- For differential equations, we called this the *superposition principle*
- More about this later

What is a Matrix?

Figure : I cannot tell you what a matrix is, I have to show you.

- A matrix is a rectangular grid of numbers
- For example

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$$

- as well as

$$\begin{pmatrix} 8 & 6 & 7 \\ 5 & 3 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 4 & 4 \end{pmatrix}$$

Shape of a Matrix

- The shape of a matrix is determined by the number of rows and columns it has
- An $m \times n$ matrix A has m rows and n columns.
- If $m = n$, then the matrix is called **square**
- For example the matrices on the previous slide where 3×3, 2×3 and 3×2, respectively.
- We may use index notation $A = (a_{ij})$ to mean that the entries of the matrix A are given by a_{ij}

Adding/Subtracting Matrices

We can **add** matrices that are the *same shape*.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 8 & 0 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 1+8 & 2+0 \\ 3+7 & 4+9 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 10 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+3 & 3+(-2) \\ 4+4 & 5+(-2) & 6+1 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 1 \\ 8 & 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \end{pmatrix} = \textbf{nonsense}.$$

Scaling Matrices

We can multiply matrices by scalars

$$7\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7\cdot 1 & 7\cdot 2 \\ 7\cdot 3 & 7\cdot 4 \end{pmatrix} = \begin{pmatrix} 7 & 14 \\ 21 & 28 \end{pmatrix}$$

$$\begin{aligned} 4\begin{pmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \end{pmatrix} &= 4\begin{pmatrix} 4\cdot 1 & 4\cdot 3 & 4\cdot(-2) \\ 4\cdot 4 & 4\cdot(-2) & 4\cdot 1 \end{pmatrix} \\ &= 4\begin{pmatrix} 4 & 12 & -8 \\ 16 & -8 & 4 \end{pmatrix} \end{aligned}$$

Multiplying Matrices

We can **multiply matrices** of compatible size

- To multiply A and B to get AB, A must have the same number of columns as B has rows

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 9 & 9 \end{pmatrix} \quad \textbf{makes sense}.$$

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 9 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \textbf{nonsense}.$$

Multiplying Matrices

- if $A = (a_{ij})$ is an $\ell \times m$ matrix
- and $B = (b_{jk})$ is an $m \times n$ matrix
- the product $AB = (c_{ik})$ is an $\ell \times n$ matrix with

$$c_{ij} = \sum_{j=0}^{m} a_{ij} b_{jk}.$$

- for example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 9 & 9 \end{pmatrix} = \begin{pmatrix} 32 & 29 \\ 71 & 62 \\ 110 & 95 \end{pmatrix}$$

Multiplying Matrices

- To show more work:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 9 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 3 + 2\cdot 1 + 3\cdot 9 & 1\cdot 2 + 2\cdot 0 + 3\cdot 9 \\ 4\cdot 3 + 5\cdot 1 + 6\cdot 9 & 4\cdot 2 + 5\cdot 0 + 6\cdot 9 \\ 7\cdot 3 + 8\cdot 1 + 9\cdot 9 & 7\cdot 2 + 8\cdot 0 + 9\cdot 9 \end{pmatrix}$$

$$= \begin{pmatrix} 32 & 29 \\ 71 & 62 \\ 110 & 95 \end{pmatrix}$$

Matrix Transpose

We can take the **transpose** of a matrix

- if $A = (a_{ij})$ is an $m \times n$ matrix
- then the transpose A^T is an $n \times m$ matrix with entries (a_{ji})
- for example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 8 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 3 & 8 \end{pmatrix}$$

Matrix Conjugate Transpose

For matrices with complex entries, we can take the **conjugate transpose**

- also called the **Hermitian conjugate**
- if $A = (a_{ij})$ is an $m \times n$ matrix with complex entries
- then the conjugate transpose A^* is an $n \times m$ matrix with entries $(\overline{a}_{ji})$
- here $\overline{a}_{ji}$ denotes the complex conjugate of a_{ij}

$$\begin{pmatrix} 1 & 2+i \\ 4-i & 5i \\ 7 & 8-2i \end{pmatrix}^* = \begin{pmatrix} 1 & 4+i & 7 \\ 2-i & -5i & 8+2i \end{pmatrix}$$

Matrix Determinant

For square matrices, we also have the notion of a **determinant**

- $\det(A)$ means the determinant of A
- for a 2×2 and 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Matrix Determinant

- what about BIGGER matrices?
- we calculate the determinant recursively ...
- if $A = (a_{ij})$ is a larger $n \times n$ matrix
- the determinant may be calculated via **row expansion**

$$\det(A) = a_{11}A_{11} - a_{12}A_{21} + a_{13}A_{31} - \cdots + (-1)^{n+1}a_{1n}A_{nn}$$

Where A_{ij} denotes the $(n-1) \times (n-1)$ **cofactor matrix** obtained from A by deleting the i'th row and j'th column

Matrix Determinant

- examples:

$$\det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 1\cdot 1 - 1\cdot(-1) = 2$$

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} = 400$$

Matrix Inverse

a Let A be a square matrix

- if $\det(A) \neq 0$, then is called **nonsingular**
- if $\det(A) = 0$, then A is **singular**
- a nonsingular square matrix A has an **inverse** A^{-1}
- the inverse is the *unique* matrix satisfying $A \cdot A^{-1} = A^{-1} \cdot A = I$
- here, I is the **identity matrix**

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Matrix Inverse

- the inverse of a 2×2 nonsingular matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- more generally, we can find the inverse of A by row reducing the $n \times 2n$ matrix $[A|I]$
- the row reduced form will be $[I|A^{-1}]$.

Matrix Inverse

- examples:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^{-1} = \textbf{does not exist} \text{ (singular matrix)}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Summary!

What we did today:

- We looked at what our class is about
- We reviewed some ideas about matrices

Plan for next time:

- Systems of Linear Algebraic Equations
- Linear Independence
- Eigenvectors and Eigenvalues