Math 309 Lecture 13

Fourier Series

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Today!

Plan for today:

- Fourier Series

Next time:

- Convergence of Fourier Series

Outline

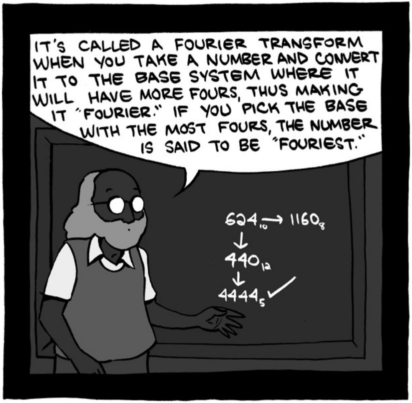
Periodic Functions

- A function $f(x)$ is **periodic** with **period** T if $f(x+T)=f(x)$ for all x
- The smallest value of T for which this holds is called the **fundamental peroiod** of $f(x)$
- NOTE: constant functions are periodic, but have no fundamental period
- Important observation:

$$1, \sin(n\pi x/L),\ \cos(n\pi x/L), n = 1, 2, 3, \dots$$

  have period $2L$ for all n.
- As does *any* linear combination

The Big Idea



Teaching math was way more fun after tenure.

- The big idea: maybe *any* periodic function of period $2L$ is some linear combination of these
- This is the idea behind Fourier series!
- Has wonderful applications in solving difficult differential equations
- Usually used in conjuction with some sort of boundary value problem

An Example: Square Wave Function

Example

Express the square wave function

$$f(x) = \begin{cases} 0, & \lfloor x \rfloor \text{ even} \\ 1, & \lfloor x \rfloor \text{ odd} \end{cases}$$

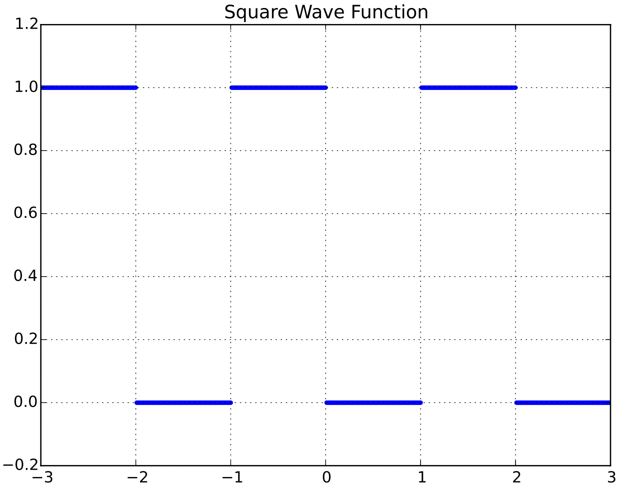
as a linear combination of sines and cosines.

In other words, we wish to find constants a_n and b_n such that

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + b_1 \sin\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \dots$$

Where L is half the fundamental period of $f(x)$ ($L = 1$).

Square Wave Function



Fourier Series

- Fourier series are like Taylor series in that we approx. a complicated function in terms of a sum of simpler functions

$$c_0 + \sum_{n=1}^{\infty} c_n (x - x_0)^n, \quad \textbf{Taylor series}$$

- Fourier series use trig functions, while Taylor series use polynomials

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \left(\frac{n\pi x}{L}\right) \right], \quad \textbf{Fourier series}$$

- The values of a_n, b_n depend on the function $f(x)$

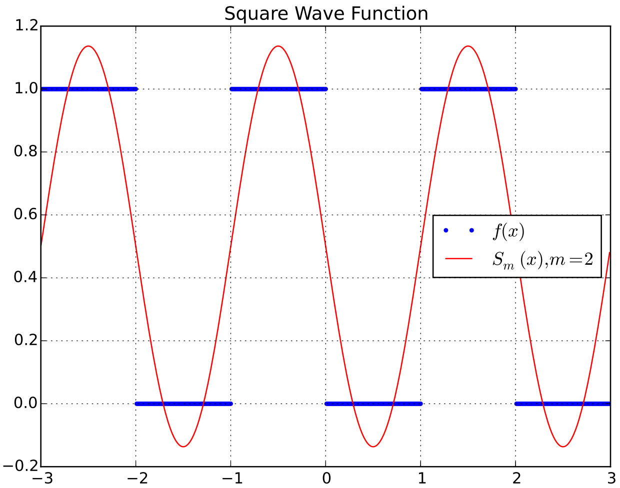
Partial Sums

- We can also think about partial sums of the Fourier series as approximations to $f(x)$ in the same way Taylor polynomials approximate a function
- The m'th **partial Fourier sum** is

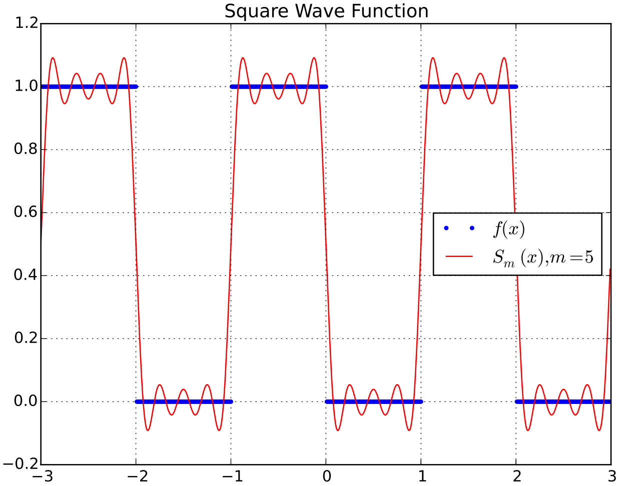
$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \left(\frac{n\pi x}{L}\right)\right]$$

- Just like Taylor polynomials, as m gets larger, $S_m(x)$ becomes a better approximation of $f(x)$
- If $f(x)$ is the square wave function from before, $a_0 = 1$, $a_n = 0$ for $n > 1$, and

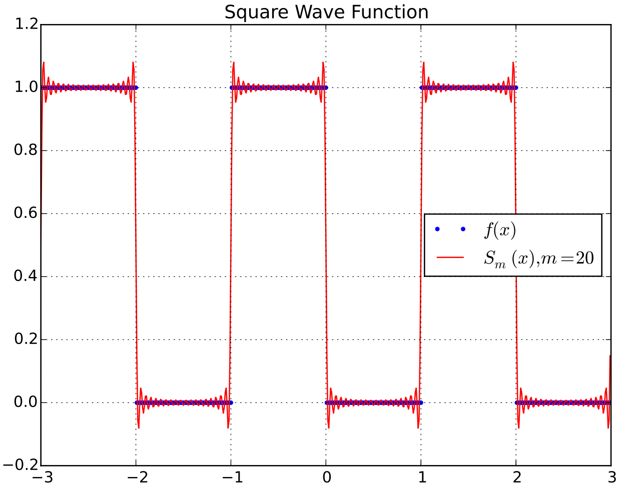
Square Wave Function Approximation $S_2(x)$



Square Wave Function Approximation $S_5(x)$



Square Wave Function Approximation $S_{20}(x)$



Fourier Series

- For the square wave function we are studying

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \left(\frac{n\pi x}{L}\right)\right]$$

- with $a_0 = 1$, $a_n = 0$ for all $n > 0$, and

$$b_n = \begin{cases} -2/(\pi n), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

- how did we arrive at these numbers?

Space of Periodic Functions

- think about the set $\mathcal{P}_T$ of functions with period $T = 2L$ as a **vector space**
- this makes sense! closed under taking linear combinations:

  if $f_1(x), f_2(x)$ have period T then $f_i(x+T) = f_i(x)$ for all x

  and therefore for all x,

$$c_1 f_1(x+T) + c_2 f_2(x+T) = c_1 f_1(x) + c_2 f_2(x)$$

  making $c_1 f_1(x) + c_2 f_2(x)$ periodic

Orthogonality

- this vector space has an inner product (dot product):
- if $f(x) \in \mathcal{P}_T, g(x) \in \mathcal{P}_T$, then

$$\langle f(x), g(x) \rangle := \int_{-T/2}^{T/2} f(x)g(x)dx$$

- $f(x), g(x) \in \mathcal{P}_T$ are called **orthogonal** if $\langle f, g \rangle = 0$
- a set of functions $\{f_1(x), f_2(x), \dots, \} \subseteq \mathcal{P}_T$ are **mutually orthogonal** if $\langle f_i, f_j \rangle = 0$ for all $i \neq j$.

Theorem

The infinite set of functions in $\mathcal{P})_{2L}$, given by

$$\left\{ \frac{1}{2}, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) : n = 1, 2, 3, \dots \right\}$$

is mutually orthogonal.

Orthogonality

Consequently, if

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \left(\frac{n\pi x}{L}\right) \right]$$

then

$$\langle f(x), \cos(m\pi x/L) \rangle = \langle a_m \cos(m\pi x/L), \cos(m\pi x/L) \rangle = a_m L.$$

This means that

$$a_m = \frac{1}{L} \langle f(x), \cos(m\pi x/L) \rangle = \frac{1}{L} \int_{-L}^{L} f(x) \cos(m\pi x/L) dx.$$

Similar equations hold for b_m and a_0.

Euler-Fourier Formulas

Theorem

Let $f(x) \in \mathcal{P}_{2L}$. Then the Fourier series for $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \left(\frac{n\pi x}{L}\right) \right]$$

has coefficients given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \;\; n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \;\; n = 1, 2, 3, \ldots$$

Square Wave Function

Question

Consider the square wave function

$$f(x) = \begin{cases} 0, & 0 \leq x < L, \\ 1, & -L \leq x < 0 \end{cases} \quad \text{with } f(x+2L) = f(x) \text{ for all } x$$

What are it's Fourier coefficients?

- We can figure this out by using the Euler-Fourier Formulas!
- Note that $f(x)$ has fundamental period $2L$

Square Wave Fourier Coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{0} \cos\left(\frac{n\pi x}{L}\right)dx \qquad = \frac{1}{L}\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \begin{cases} 1, & n = 0 \\ 0, & n > 0 \end{cases} \qquad = \frac{-1}{n\pi}(\cos(0) - \cos(-n\pi))$$

$$= \begin{cases} -2/(n\pi), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Fourier Series for a Square Wave

- this agrees with the coefficients we stated earlier
- the Fourier series for $f(x)$ is therefore

$$f(x) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi x}{L}\right).$$

Summary!

What we did today:

- Fourier series

Plan for next time:

- Convergence of Fourier Series