Math 309 Lecture 14

Convergence of Fourier Series

W.R. Casper

Department of Mathematics
University of Washington

November 4, 2015

Today!

Plan for today:

- More Fourier Series Examples
- Convergence of Fourier Series

Next time:

- Even and Odd Functions

Outline

The First Example

Question

Consider the function

$$f(x) = e^x, \ -L < x < L, \text{ with } f(x+2L) = f(x) \text{ for all } x$$

What is its Fourier series?

- the fundamental period of the function is $2L$
- we use the Euler-Cauchy formulas to determine its Fourier series

Fourier Coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} e^x\cos\left(\frac{n\pi x}{L}\right)dx \quad = \frac{1}{L}\int_{-L}^{L} e^x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}(e^L - e^{-L})\frac{(-1)^n}{1+(n\pi/L)^2} \quad = \frac{-1}{L}(e^L - e^{-L})\frac{(-1)^n(n\pi/L)}{1+(n\pi/L)^2}$$

The Second Example

Question

Consider the function

$$f(x) = 1 - |x|, \; -1 < x < 1, \; \text{with } f(x+2) = f(x) \text{ for all } x$$

What is its Fourier series?

- the fundamental period of the function is 2
- $f(x)$ is even around 0, so $b_n = 0$ for all n
- we use the Euler-Cauchy formulas to determine a_n

Fourier Coefficients

$$
\begin{aligned}
a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)dx \\
&= 2\int_0^1 f(x)\cos(\pi n x)dx, \quad \text{(since } f(x) \text{ is even)} \\
&= 2\int_0^1 (1-x)\cos(\pi n x)dx \\
&= \frac{2}{\pi n}\int_0^1 \sin(\pi n x)dx, \quad \text{(integration by parts)} \\
&= -\frac{2}{\pi^2 n^2}((-1)^n - 1) = \begin{cases} 4/(\pi^2 n^2), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
\end{aligned}
$$

Fourier Series

- this works unless $n = 0$, for which we calculate $a_0 = 1$
- the Fourier series for $f(x)$ is therefore

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{4}{\pi^2(2n+1)^2} \cos(\pi(2n+1)x).$$

Convergence of a Series

Question

Suppose that $f(x)$ is a periodic function, and that a_m and b_m are the coefficients given by the Euler-Fourier formulas. When does the Fourier series

$$\frac{a_0}{2} + \sum_{m=0}^{\infty} \left[a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right]$$

converge to the function $f(x)$.

- this is a big question – with many different answers
- could spend all quarter thinking about just this
- with **any** question about convergence, always ask **to what** and **in what sense**

Pointwise Convergence

- in this class, by convergence, we will mean **pointwise convergence**, ie. we are asking for which fixed values of x the infinite sum

$$\frac{a_0}{2} + \sum_{m=0}^{\infty} \left[a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right]$$

  is equal to $f(x)$.
- for an arbitrary function, this is a complicated question
- we will focus instead on functions which are nice enough to give us a nice answer

Convergence Theorem

- let $f(x)$ be a function, and let $f(a-)$ and $f(a+)$ denote the limit of $f(x)$ as x approaches a from the left or right, respectively.
- recall that a function is **piecewise-continuous** in a closed interval $[a, b]$ if for all values of x the limits $f(x-)$ and $f(x+)$ exist and are finite, and that $f(x-)$ and $f(x+)$ differ at only finitely many values of x in $[a, b]$

Theorem

Suppose that $f(x)$ is periodic with fundamental period $2L$, and that $f(x)$ and $f'(x)$ are piecewise-continuous in $-L \leq x < L$. Then the Fourier series of $f(x)$ converges pointwise to $(f(x+) + f(x-))/2$.

Interpretation

- Suppose that $f(x)$ satisfies the assumptions of the theorem
- if $f(x)$ is continuous at a, then $f(a+) = f(a-) = f(a)$ and therefore the Fourier series converges evaluated at $x = a$ converges to $f(a)$
- if $f(x)$ is discontinuous at a, then the Fourier series converges to the *average* of the left and right limits of $f(x)$ as $x \to a$.
- for example, if $f(x)$ is the square wave function

$$f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & -1 \leq x < 0 \end{cases}, \quad f(x+2) = f(x) \text{ for all } x.$$

- , then the Fourier series converges to $f(x)$ everywhere except for the discontinuous points, where it converges to the average of 0 and 1, ie. $1/2$.