Math 309 Lecture 2

Linear Algebraic Systems and Eigenstuff

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Today!

Plan for today:

- Systems of Linear Algebraic Equations
- Linear Independence
- Eigenvectors and Eigenvalues

Next time:

- More linear algebra review

Outline

Algebraic Systems of Equations

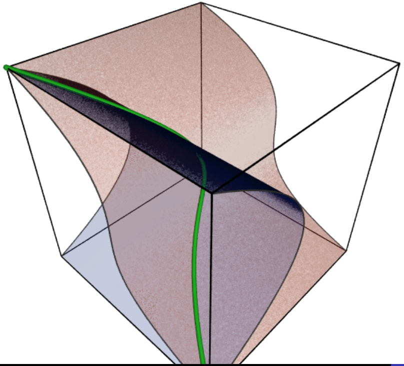
- An algebraic system of equations is something of the form

$$\begin{cases} F_1(x_1, x_2, \dots, x_n) & = 0 \\ F_2(x_1, x_2, \dots, x_n) & = 0 \\ \vdots & = \vdots \\ F_m(x_1, x_2, \dots, x_n) & = 0 \end{cases}$$

- $x_1, \dots, x_n$ are **variables**
- $F_1, \dots, F_m$ are functions describing relationships between variables
- **solutions** are values of $x_1, \dots, x_n$ satisfying relationships

Algebraic System Example

Figure : Graph of solutions to the system



- For example

$$\begin{cases} xz - y^2 & = 0 \\ y - z^2 & = 0 \end{cases}$$

- Solution is green curve
- Made from intersection of surfaces
- **Never forget!** solutions to algebraic systems have **both** *algebraic* and *geometric* meaning

Linear Algebraic Systems

- An algebraic system is **linear** if it is of the form

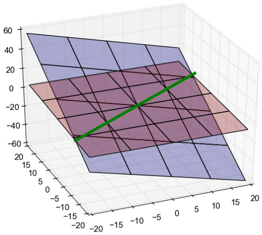
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 & = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 & = 0 \\ \vdots & = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m & = 0 \end{cases}$$

- for some constants a_{ij} and b_i
- in terms of matrices: $A\vec{x} = \vec{b}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Linear Algebraic System Example

Figure : Graph of solutions to the system



- For example

$$\begin{cases} \phantom{x - {}} 2y - 8z & = 0 \\ x - 2y + z & = 0 \end{cases}$$

- Matrix version:

$$\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Solution is green curve
- Made from intersection of planes
- linear equations make **straight** things

Solving Linear Systems

Question

How can we algebraically solve a linear system?

- we can use **Gaussian elimination**
- given a linear system $A\vec{x} = \vec{b}$ as above
- form augmented matrix $[A|\vec{b}]$:

$$[A|\vec{b}] = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

- We perform **elementary row operations** to put $[A|\vec{b}]$ in **row reduced echelon form** (RREF)

Example Solution 1

- consider the previous example linear system

$$\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- the augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 2 & 8 & 0 \\ 1 & 2 & 1 & 0 \end{array} \right)$$

- row reduce

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 8 & 0 \end{array} \right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 2 & 8 & 0 \end{array} \right) \xrightarrow{R_2/2} \left(\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right)$$

Example Solution 1

- how do we interpret RREF?

$$\left(\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right)$$

- first nonzero entry of a row is a **pivot** – corresponding column is a **pivot column**
- pivot columns correspond to **dependent variables**
- other columns correspond to **free variables**
- express dependent variables in terms of free variables
- row 1 says $x - 7z = 0$
- row 2 says $y + 4z = 0$
- solution is

$$x = 7z, \quad y = -4z.$$

Example Solution 1

- consider the linear system

$$\begin{pmatrix} 2 & 6 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

- the augmented matrix is

$$\left(\begin{array}{cc|c} 2 & 6 & 2 \\ 3 & -1 & -2 \end{array}\right)$$

- row reduce

$$\xrightarrow{R_1/2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & -1 & -2 \end{array}\right) \xrightarrow{R_2-3R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -7 & -5 \end{array}\right)$$

$$\xrightarrow{R_2/(-7)} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 5/7 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{cc|c} 1 & 0 & -3/7 \\ 0 & 1 & 5/7 \end{array}\right)$$

- solution is $x = -3/7$, $y = 5/7$

Vectors

- (column) **vectors** are $m \times 1$ matrices
- vectors are describe things with magnitude and direction
- like matrices, we can add vectors and multiply vectors by scalars
- we *cannot* multiply vectors (shapes are not compatible)
- given vectors $\vec{v}_1, \ldots, \vec{v}_n$ we can make a new vector by taking a **linear combination**:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

Linear Independence

- a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is called **linearly dependent** if there exist constants $c_1, \ldots, c_n$ not all zero, so that
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$
- a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is called **linearly independent** if the only linear combination satisfying
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$
is the **trivial** linear combination $c_1 = 0, c_2 = 0, \ldots c_n = 0$.

Linear Independence Example

- Given vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

- is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent?
- no, since

$$-1\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + -1\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear Independence Example

- Given vectors

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

- is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent?
- yes (we will show this in a second)

Checking Linear Independence

Question

How do we tell if a set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent?

- we are trying to decide if there exist $c_1, \dots, c_n$ such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

- in terms of matrices, trying to solve $V\vec{c} = \vec{0}$ for

$$V = (\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n), \ \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \ddots \\ c_n \end{pmatrix}$$

Checking Linear Independence Example

- consider the vectors

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

- $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ gives:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- the augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$$

Checking Linear Independence Example

- we row reduce

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3/(-2)} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

- therefore the *only* solution is $c_1 = 0, c_2 = 0, c_3 = 0$
- this means that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent

Checking Linear Independence

- recall that the **rank** of a matrix A is the number of **pivot columns** in its RREF
- to decide linear independence, the following theorem is helpful:

Theorem

The set of vectors

$$\{\vec{v}_1, \ldots, \vec{v}_n\}$$

is linearly indepdendent if and only if the associated matrix $V = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_n)$ has rank n

- therefore we can check for linear dependence by calculating the rank of the corresponding matrix

What are Eigenvectors?

Figure : *eigen* is German for *proper*

- Let A be an $n \times n$ matrix
- An **eigenvector** $\vec{v}$ of A with **eigenvalue** λ is a *nonzero* vector $\vec{v}$ satisfying

$$A\vec{v} = \lambda\vec{v}$$

- For example $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with eigenvalue 2

Finding Eigenvalues

Question

How can we figure out what eigenvalues a matrix has?

- look at the **characteristic polynomial**

$$p_A(x) = \det(A - xI)$$

- eigenvalues of A are roots of the characteristic polynomial
- for example, consider:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- $p_A(x) = \det(A - xI) = x^2 - 2x$
- eigenvalues are 0 and 2

Finding Eigenvectors

Question

How can we figure out what eigenvectors a matrix has?

- given eigenvalue λ, solve the system $A\vec{v} = \lambda\vec{v}$
- equivalently, solve the system $(A - \lambda I)\vec{v} = \vec{0}$
- Note: the solutions to $(A - \lambda I)\vec{v} = \vec{0}$ form a **vector space**, called the **eigenspace** of A for λ
- denoted $E_\lambda(A)$

Finding Eigenvectors

- for example, consider:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- the eigenvalues are again $0, 2$
- the eigenspaces are obtained by solving $A\vec{v} = \vec{0}$ and $(A - 2I)\vec{v} = \vec{0}$
- this gives

$$E_0(A) = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : x \in \mathbb{C} \right\}$$

$$E_2(A) = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{C} \right\}$$

Summary!

What we did today:

- Systems of Linear Algebraic Equations
- Linear Independence
- Eigenvectors and Eigenvalues

Plan for next time:

- More on eigenvectors and eigenvalues
- More linear algebra stuff in general