Math 309 Lecture 3

More Eigenthings

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Today!

Plan for today:

- Eigenvector and Eigenvalue Practice
- Matrices as Maps
- Eigenspace Decomposition and Diagonalization

Next time:

- First order Linear Systems of Equations

Outline

Eigenreview!

- let A be an $n \times n$ matrix
- a vector $\vec{v}$ is an **eigenvector** with **eigenvalue** λ if

$$\vec{v} \neq \vec{0}, \quad \text{and} \quad A\vec{v} = \lambda\vec{v}$$

- e.g. $\vec{v} \neq \vec{0}$ and $(A - \lambda I)\vec{v} = 0$
- define the **eigenspace** of λ:

$$E_\lambda(A) := \{\vec{v} : A\vec{v} = \lambda\vec{v}\}$$

- it's a vector space!!! (the nullspace of the matrix $A - \lambda I$)

When is $E_\lambda(A) \neq \{0\}$?

- λ is an eigenvalue of A if $E_A(\lambda) \neq \{0\}$
- for which values of λ does this happen?
- recall the **nullspace** of B is $\mathcal{N}(B) = \{\vec{v} : B\vec{v} = \vec{0}\}$

$$B \text{ nonsingular} \;\Leftrightarrow\; \mathcal{N}(B) = \{0\}$$

$$B \text{ nonsingular} \;\Leftrightarrow\; \det(B) \neq 0$$

- therefore $\mathcal{N}(B) = \{0\} \Leftrightarrow \det(B) \neq 0$
- since $E_\lambda(A) = \mathcal{N}(A - \lambda I)$, we see:

$$E_\lambda(A) \neq \{0\} \Leftrightarrow \det(A - \lambda I) = 0$$

Finding Eigenvalues

- we define the **characteristic polynomial** of A:

$$p_A(x) = \det(A - xI)$$

- eigenvalues of A are roots of the characteristic polynomial
- for example, consider:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

- $p_A(x) = \det(A - xI) = x^2 - 2x - 1$
- eigenvalues are $1 \pm \sqrt{2}$

Finding Eigenvectors

- what are the corresponding eigenspaces of

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

- need to calculate nullspaces $\mathcal{N}(A - 1 \pm \sqrt{2})$
- we know how to do this! (RREF):

$$E_{1+\sqrt{2}}(A) = \mathcal{N}(A - (1 + \sqrt{2})I) = \text{span}\left\{\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\right\}$$

$$E_{1-\sqrt{2}}(A) = \mathcal{N}(A - (1 - \sqrt{2})I) = \text{span}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\}$$

Functions

- a **function** f from $\mathbb{R}^n$ to $\mathbb{R}^m$
- takes in an n-vector $\vec{v}$
- returns an m-vector $f(\vec{v})$
- denote this by $f : \mathbb{R}^n \to \mathbb{R}^m$
- example: $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f\left(\begin{pmatrix} \theta \\ \phi \end{pmatrix}\right) = \begin{pmatrix} \cos(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\phi) \end{pmatrix}$$

- takes $\mathbb{R}^2$ to a sphere in $\mathbb{R}^3$

Linear Functions

- a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if it respects addition and scalar multiplication, ie.

$$f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w}) \text{ and } f(c\vec{v}) = cf(\vec{v})$$

- for example:

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 3y \\ 3x - 4y \end{pmatrix}$$

  is linear

-
$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ xy \end{pmatrix}$$

  is not linear

Matrices Define Linear Functions

- let A be an $m \times n$ matrix
- define $f_A : \mathbb{R}^n \to \mathbb{R}^m$ by $f_A(\vec{v}) = A\vec{v}$
- then f is a linear function
- for example:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$$

$$f_A\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 3x - 4y \end{pmatrix}$$

Linear Functions Define Matrices

- any linear function f is of the form f_A for some matrix A

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Then $f = f_A$ for A the $m \times n$ matrix

$$A = (f(\vec{e}_1)\ f(\vec{e}_2)\ \ldots\ f(\vec{e}_n)).$$

- here $\vec{e}_1, \ldots, \vec{e}_n$ are the **standard basis vectors** for $\mathbb{R}^n$
- e.g. $I = (\vec{e}_1\ \vec{e}_2\ \ldots\ \vec{e}_n)$
- thus studying linear functions is the *same thing* as studying matrices

Transform the Earth!

- we can visualize $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by a 2×2 matrix A
- for example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad f = f_A : \vec{v} \mapsto A\vec{v}$$

What Happened to Earth?

- the earth got stretched out!
- roughly twice as wide in stretch direction
- stretch direction is 45 degrees counter-clockwise from positive x-axis
- explained by eigenvectors/eigenvalues!
- eigenvalues: $1, 2$
- eigenspaces:

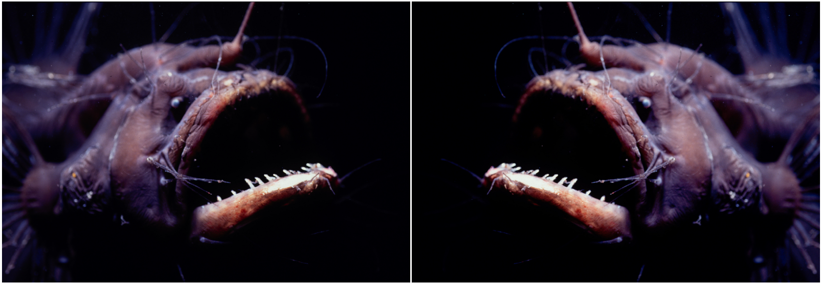
$$E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

- eigenvectors of eigenvalue 2 **point in stretch direction**!

Transform the Anglerfish!

- another example

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f = f_A : \vec{v} \mapsto A\vec{v}$$



What Happened to our Fish?

- we flipped our fish in the x-direction
- eigenvalue explanation?
- eigenvalues are 1 and -1
- eigenspaces:

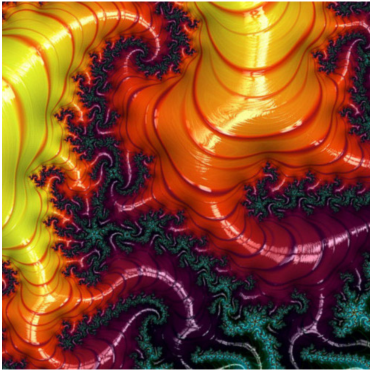
$$E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E_1 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

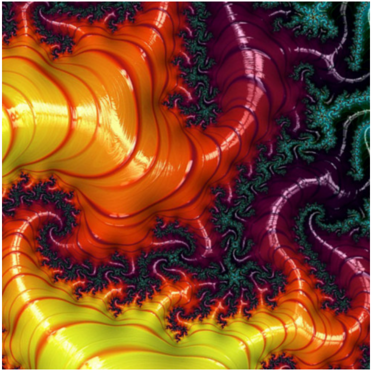
- eigenvector for eigenvalue -1 in x-direction!

Transform the Fractal!

- another example

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f = f_A : \vec{v} \mapsto A\vec{v}$$





What Happened to our Fractal?

- we rotated counter-clockwise 90 degrees
- eigenvalue explanation?
- eigenvalues are i and $-i$
- eigenspaces:

$$E_i = \text{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}, \quad E_{-i} = \text{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$$

- rotation gives us complex eigenvalues!

Summary: Eigenvectors/Eigenvalues tell a Story

Figure: If the Fonz were an eigenvector, he would have eigenvalue *aaaaaaaaay!*

- magnitude of eigenvalue determines dilation/contraction (scaling)
- direction of eigenvector determines scaling direction
- negative and complex eigenvalues determine rotation and reflection
- direction of eigenvector determines reflection direction

Diagonalizable Matrices

- a matrix D is **diagonal** if the only nonzero entries are on the main diagonal
- for example:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$$

  is diagonal.
- a matrix A is **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D satisfying

$$P^{-1}AP = D.$$

Diagonalizable Matrices and Eigenstuff

- how can we find P and D for a matrix A?
- the diagonal entries of D are the eigenvalues of A
- the column vectors of P are the corresponding eigenvectors
- this tells us *how* to diagonalize a matrix: find its eigenvectors and eigenvalues

Diagonalizing Matrices Example

- Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

- The eigenvalues of A, are 1 and 2
- The eigenspaces of A are

$$E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

- Define

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

- then one may check $P^{-1}AP = D$

Eigenbasis for $\mathbb{R}^n$

- **important**: not all matrices are diagonalizable!
- example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  is NOT diagonalizable
- an $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a **eigenbasis**
- e.g. $\mathbb{R}^n$ has a basis consisting of eigenvectors of A
- how can we tell?

Eigenvalue Multiplicity

- the **algebraic multiplicity** of an eigenvalue λ of A is the number of times it is a root of the characteristic polynomial $p_A(x)$
- the **geometric multiplicity** of an eigenvalue λ is the dimension of the eigenspace $E_\lambda(A)$
- $\mathbb{R}^n$ has an eigenbasis if and only if the sum of the geometric multiplicities of eigenvalues of A is 1

Theorem

The algebraic multiplicity of an eigenvalue is always $\geq$ the geometric multiplicity

Corollary

If all the eigenvalues of A have multiplicity 1, then A is diagonalizable.

Normality

- let $A^\dagger$ denote the Hermitian conjugate of A
- two square matrices A and B **commute** if $AB = BA$
- a matrix A is called **normal** if A and $A^\dagger$ commute
- a matrix U is called **unitary** if $U^\dagger = U^{-1}$

Theorem (Spectral Theorem)

Let A be an $n \times n$ square matrix. The following are equivalent

(a) A is normal

(b) there exists a unitary matrix U and diagonal matrix D with $U^{-1}AU = D$

- in particular, normal matrices are diagonalizable

Example

- for example, consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- the hermitian conjugate is

$$A^{\dagger} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

- $$AA^{\dagger} - A^{\dagger}A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- therefore by the spectral theorem A is not diagonalizable

Summary!

What we did today:

- Systems of Linear Algebraic Equations
- Linear Independence
- Eigenvectors and Eigenvalues

Plan for next time:

- Systems of first order ODEs