Math 309 Lecture 4

More Eigenthings

W.R. Casper

Department of Mathematics
University of Washington

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Today!

Plan for today:

- First Order Linear Systems of ODEs
- Homogeneous First Order Linear Systems with Constant Coefficients

Next time:

- More Homogeneous First Order Linear Systems

Outline

1. First Order Linear Systems of ODEs
   - Basics
   - Superposition Principle
   - Example
2. Homogeneous Linear Systems with Constant Coefficients
   - Using Eigenvalues to Solve
   - Example

First Order Linear Systems

- a **first order linear system** of equations is something of the form

$$\begin{aligned}
y_1'(t) &= a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + \dots a_{1n}(t)y_n(t) + b_1(t)\\
y_2'(t) &= a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + \dots a_{2n}(t)y_n(t) + b_2(t)\\
\vdots &= \vdots\\
y_n'(t) &= a_{n1}(t)y_1(t) + a_{n2}(t)y_2(t) + \dots a_{nn}(t)y_n(t) + b_n(t)
\end{aligned}$$

- where the $a_{ij}(t)$ and $b_i(t)$ are some specified functions
- and the y_i are some unknown functions we wish to find
- in terms of matrices:

$$\vec{y}'(t) = A(t)\vec{y}(t) + \vec{b}(t)$$

Homogeneous Systems

- as for algebraic linear systems, it is convenient to consider the case when $\vec{b}(t) = \vec{0}$
- a **homogeneous first order linear system** of equations is something of the form

$$y'(t) = A(t)y(t).$$

- as in the $n = 1$ case, to any first order linear system, we associate a homogeneous system

$$y'(t) = A(t)y(t) + b(t) \longrightarrow y_h'(t) = A(t)y_h(t).$$

- solving the homogeneous system will be the key to solving the full system

Superposition Principle

- just like homogeneous algebraic systems, we have a **superposition principle**

Theorem (Superposition Principle)

Suppose that $\vec{y}(t) = \vec{w}(t)$ and $\vec{y}(t) = \vec{z}(t)$ are two solutions to $y' = A(t)\vec{y}(t)$. Then for any scalars c_1, c_2, $y = c_1 w(t) + c_2 z(t)$ is also a solution.

- the set of solutions to $y' = A(t)\vec{y}(t)$ forms a *vector space*
- natural question: what is the dimension of the space of solutions?

Fundamental Set of Solutions

Theorem (Existence and Uniqueness)

Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix, and $\vec{b}(t) = (b_i(t))$ be a vector with $a_{ij}(t), b_i(t)$ continuous on the interval (α, β) for all i, j. Then for any vector $\vec{v} \in \mathbb{R}^n$ and $t_0 \in (\alpha, \beta)$, there exists a unique solution to the initial value problem

$$y'(t) = A(t)y(t) + b(t), \ \ y(t_0) = \vec{v}.$$

Corollary

If $A(t)$ is continuous on (α, β), then the set of solutions to $y'(t) = A(t)\vec{y}(t)$ on (α, β) is n-dimensional.

Fundamental Set of Solutions

- let $A(t)$ be an $n \times n$ matrix continuous on (α, β). Then a basis for the set of solutions to $y'(x) = A(x)y(x)$ is called a **fundamental set of solutions** for the system on the interval (α, β)
- in other words, a fundamental set of solutions is a set of n solutions $\vec{y}_1(x), \dots, \vec{y}_n(x)$ which are linearly independent and such that every solution to $y'(x) = A(x)y(x)$ is of the form
$$c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + \cdots + c_n\vec{y}_n(x)$$
for some constants $c_1, c_2, \dots, c_n$
- traditionally, we then call
$$y = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + \cdots + c_n\vec{y}_n(x)$$
the **general solution**

Checking Linear Independence

Proposition

If $A(t)$ is continuous on (α, β), then a set of n solutions $\vec{y}_1(x), \dots, \vec{y}_n(x)$ is a fundamental set of solutions for $y'(t) = A(t)y(t)$ in (α, β) if and only if $\vec{y}_1, \dots, \vec{y}_n$ are linearly independent.

- this is because we already know the dimension of the solution space!
- if we get n linearly independent solutions, then we get span for free
- how can we tell if the vector functions are linearly independent?

Wronskian

Theorem

A set of solutions $\vec{y}_1(x), \dots, \vec{y}_n(x)$ to $y'(t) = A(t)y(t)$ is linearly independent if and only if

$$W[\vec{y}_1(x), \dots, \vec{y}_n(x)] := \det(\vec{y}_1 \ \vec{y}_2 \ \dots \ \vec{y}_n)$$

is nonzero on some point of the interval (α, β)

- we call $W[\vec{y}_1(x), \dots, \vec{y}_n(x)]$ the **Wronskian**

An Example

Consider the homogeneous linear system of ODE's

$$y'(x) = A(x)y(x), \quad A(x) = \begin{pmatrix} 1 & -2e^{-3x} \\ 0 & 2 \end{pmatrix}$$

Two solutions on the interval $(-\infty, \infty)$ are

$$\vec{y}_1(x) = \begin{pmatrix} e^{x} \\ 0 \end{pmatrix}, \quad \vec{y}_2(x) = \begin{pmatrix} e^{-x} \\ e^{2x} \end{pmatrix}$$

Is this a fundamental set of solutions on $(-\infty, \infty)$?

An Example

We check the Wronskian!

$$W[\vec{y}_1, \vec{y}_2] = \det(\vec{y}_1 \ \vec{y}_2) = \det \begin{pmatrix} e^x & e^{-x} \\ 0 & e^{2x} \end{pmatrix} = e^{3x}$$

- it is nonzero on all of (α, β)
- consequently $\vec{y}_1$ and $\vec{y}_2$ are linearly independent
- since A is a 2×2 matrix, this means $\vec{y}_1, \vec{y}_2$ form a fundamental set of solutions on the interval $(-\infty, \infty)$
- general solution is therefore

$$\vec{y} = c_1 \begin{pmatrix} e^x \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} e^{-x} \\ e^{2x} \end{pmatrix}$$

Definition

- a **homogeneous first order linear system of equations with constant coefficients** is something of the form

$$\vec{y}(t)' = A\vec{y}(t)$$

  where A is a *constant* matrix (does not depend on t)
- can be solved explicitly by hand!
- the secret sauce to do this is eigenvalues

Using Eigenvalues

- we propose a solution of the form $\vec{y}(t) = e^{rt}\vec{v}$ for some constant vector $\vec{v}$ and some constant r
- putting this into our equation, we get

$$re^{rt}\vec{v} = \vec{y}' = A\vec{y} = Ae^{rt}\vec{v} = e^{rt}A\vec{v}$$

- dividing by e^{rt}, this says $r\vec{v} = A\vec{v}$, eg. v is an eigenvector of A with eigenvalue r

Proposition

Let A be an $n \times n$ matrix. If $\vec{v}$ is an eigenvector of A with eigenvalue r, then $\vec{y} = e^{rt}\vec{v}$ is a solution to $y' = Ay$

Question

Find the general solution of the differential equation

$$y'(t) = Ay(t), \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

- idea: use the solutions determined by eigenvectors and eigenvalues!
- eigenvalues of A are 1 and 2 (why?)
- corresponding eigenspaces:

$$E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

Example

- this gives us solutions

$$\vec{y}_1 = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{y}_2 = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- are they linearly independent?

$$W[\vec{y}_1, \vec{y}_2] = \det \begin{pmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{pmatrix} = e^{3t} \neq 0.$$

- they are linearly independent, and therefore are a fundamental solution set on $(-\infty, \infty)$
- general solution:

$$y = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Summary!

What we did today:

- Systems of first order linear ODEs

Plan for next time:

- Systems of first order ODEs