Math 309 Lecture 5

Constant Coefficient Homogeneous Linear Systems of ODEs

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Today!

Plan for today:

- Direction Fields
- Complex Eigenvalues

Next time:

- Repeated Eigenvalues
- Matrix Exponentials
- Fundamental Matrix

Outline

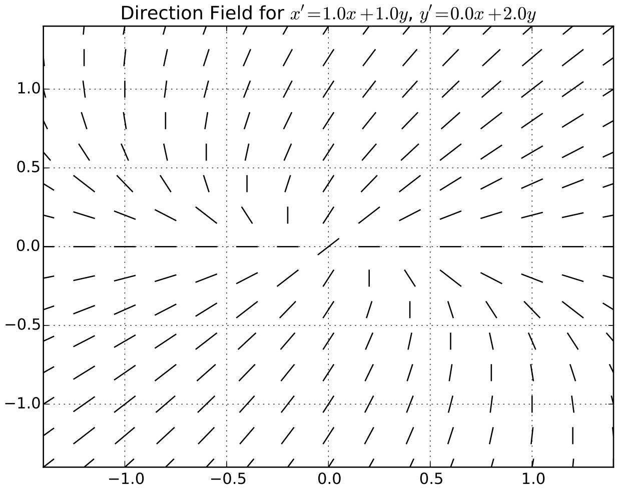
Direction Fields

Consider a 2×2 system

$$\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned}$$

- solving this system has both algebraic and geometric interpretations
- we can draw a “picture" of the equation in the **phase plane**
- here by **phase plane** we mean the y_1, y_2 plane
- strategy: at each point (y_1, y_2) draw a dash in direction of vector
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix}.$$
- result is called a **direction field**

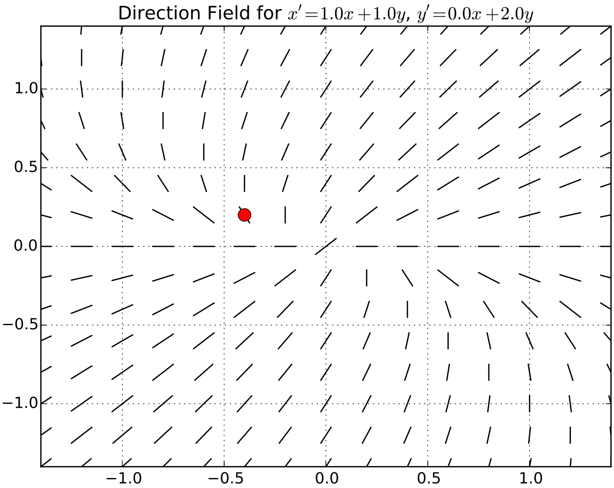
Example Direction Field



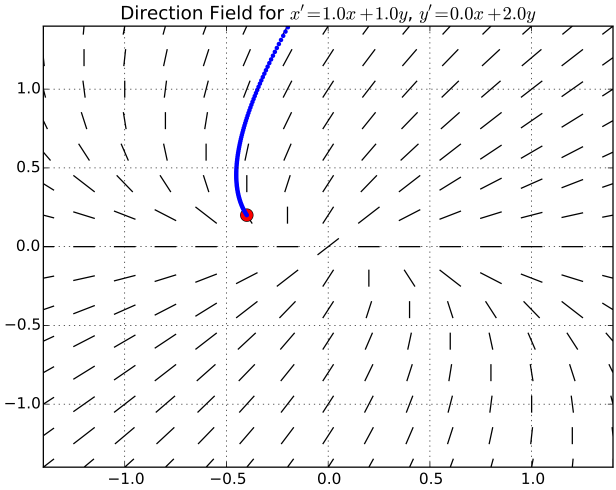
Solutions as Tangent Curves

- think of slope field as current in the ocean
- solutions to the system of equation are traced out by path of a (slow) boat
- the path a boat takes traces a curve whose tangent lines always point in direction of local slope field

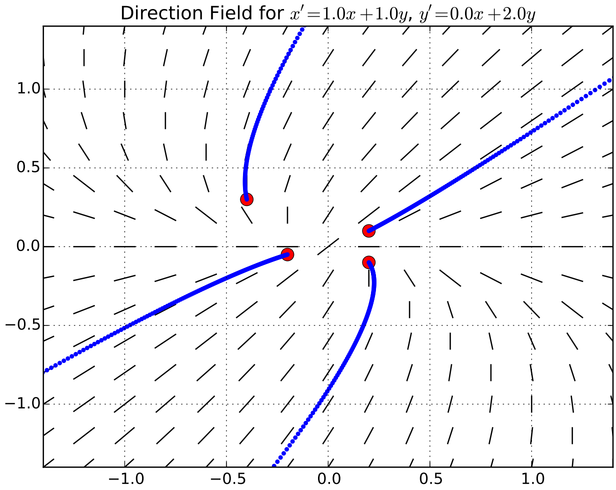
A Boat in the Ocean



The Path of the Boat



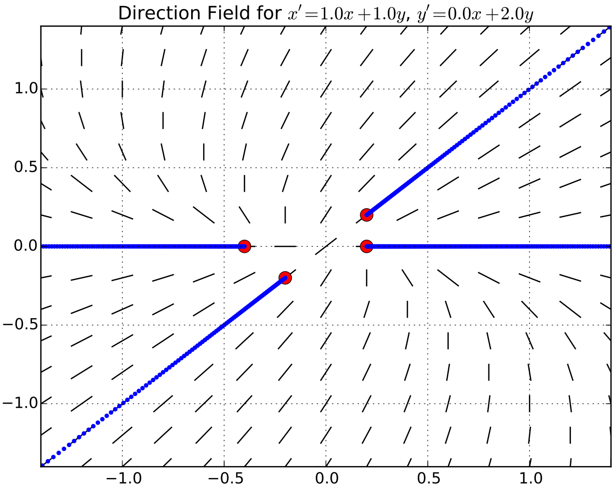
More Possible Paths



Observations:

- regardless of the initial position, the “boat" moves away from the origin
- unless if the boat starts at the origin, in which case it stays there
- for this reason, in this case we call the origin an **exponentially unstable node**
- note that there are also two straight paths the boat can take – corresponding to eigenvectors!

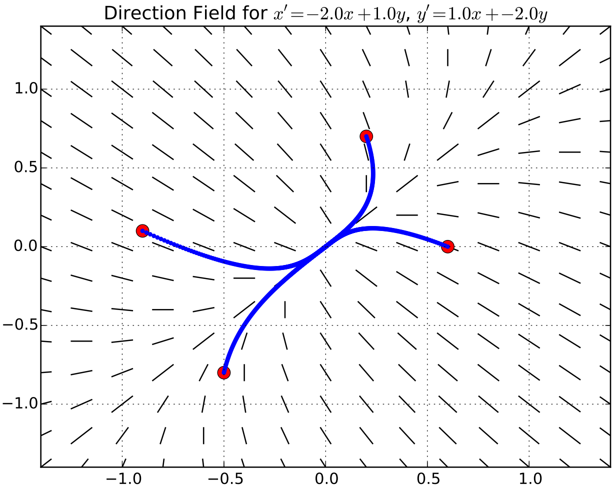
Straight Paths from Eigenvectors



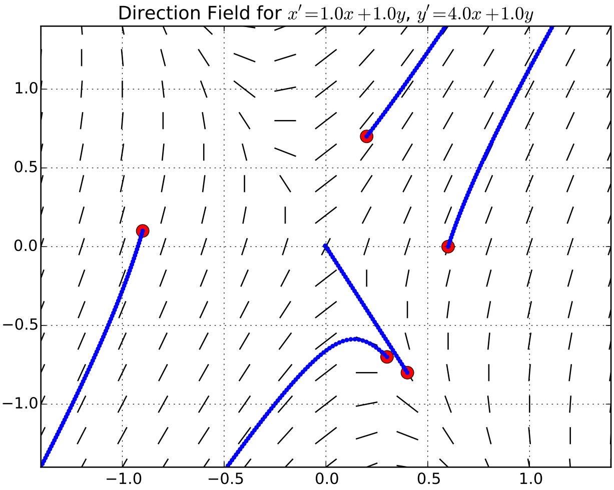
Saddle Point or Node

- the origin does not have to be an exponentially unstable node
- it may also be a **exponentially stable node** or a **saddle point**
- for an exponentially stable node, solutions tend toward the origin
- for a saddle point, solutions tend both toward and away from the origin, based on the initial condition
- for the equation $\vec{y}'(t) = A\vec{y}(t)$, the behavior of solutions around the origin depends on the *eigenvalues* of A

Exponentially Stable Node



Saddle Node


Direction Field for $x' = 1.0x + 1.0y$, $y' = 4.0x + 1.0y$
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Behavior of the Origin

- the origin is *always* a fixed point of $\vec{y}'(t) = Ay(t)$
- eg. $\vec{y}(t) = \vec{0}$ is a constant solution of the equation
- how other solutions behave is based on the eigenvalues of A:

(a) if both eigenvalues of A are real and positive, then origin is an exponentially unstable node
(b) if both eigenvalues of A are real and negative, then origin is an exponentially stable node
(c) if both eigenvalues of A are mixed sign, then origin is a saddle point
(d) what about when the eigenvalues of A are complex?

Spirally Slope Fields

- slope fields for complex eigenvalues are characterized by spiral patterns
- for example:

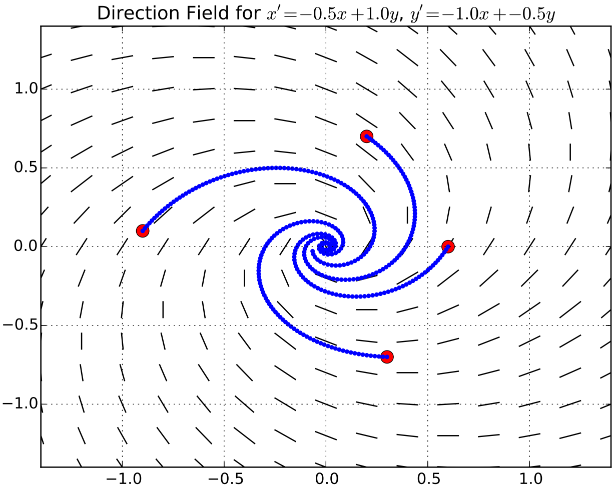
$$A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}$$

- characteristic polynomial is

$$p_A(x) = \det(A - xI) = x^2 + x + \frac{5}{4}$$

- eigenvalues of A are $-(1/2) \pm i$

Complex Eigenvalues: $-(1/2) \pm i$

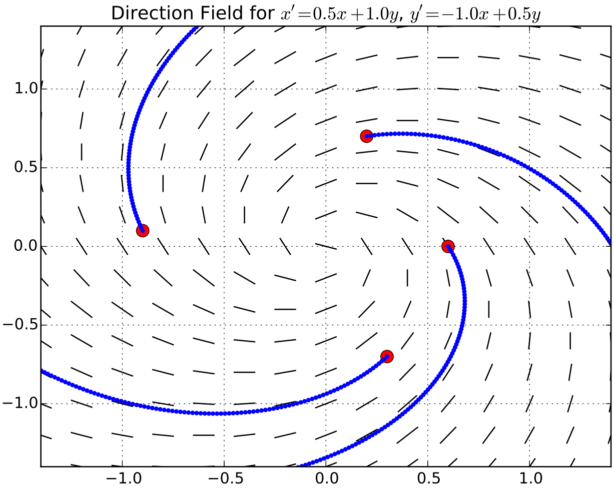


Behavior of the Origin

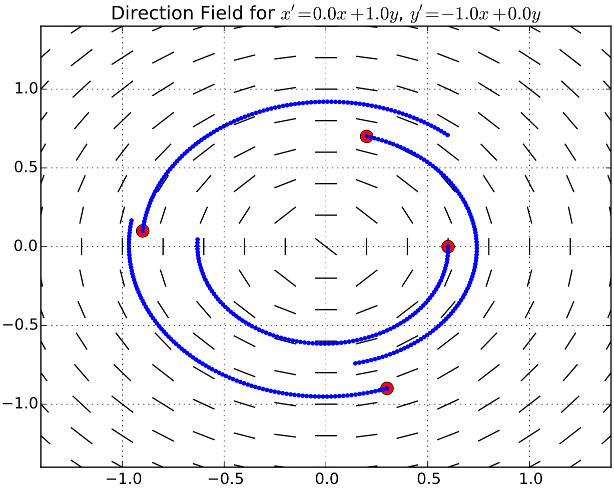
- suppose that A has complex eigenvalues
- they come in conjugate pairs! $\lambda_1 = a + ib$, $\lambda_2 = a - ib$
- the origin is *always* a fixed point of $\vec{y}'(t) = Ay(t)$
- whether our ship moves toward or away depends on value of a

(a) if a is positive, move away
(b) if a is negative, move toward
(c) if a is zero, circle around

Complex Eigenvalues: $(1/2) \pm i$



Complex Eigenvalues: $\pm i$



What about General Solutions?

Question

How do we find the general solution in the case that A has complex eigenvalues?

- use Euler's definition!

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

- we can then take our eigenvalue solutions and write them as linear combinations of *real* solutions

Example

Question

Find the general solution of the equation

$$\vec{y}'(x) = A\vec{y}, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

- first we find the eigenvalues: $-(1/2) \pm i$
- then we find the corresponding eigenspaces:

$$E_{-(1/2)+i} = \text{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\} \quad E_{-(1/2)-i} = \text{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$$

Example

- from this we get two (complex) solutions

$$\vec{y}_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t} \quad \vec{y}_2(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t}$$

- by the superposition principal we get the family of solutions:

$$\begin{aligned}
\vec{y}(t) &= c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t} \\
&= c_1 e^{-t/2} \begin{pmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \cos(t) - i\sin(t) \\ -i\cos(t) - \sin(t) \end{pmatrix} \\
&= e^{-t/2} \begin{pmatrix} (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t) \\ i(c_1 - c_2)\cos(t) - (c_1 + c_2)\sin(t) \end{pmatrix} \\
&= e^{-t/2} \begin{pmatrix} b_1\cos(t) + b_2\sin(t) \\ b_2\cos(t) - b_1\sin(t) \end{pmatrix} \\
&= b_1 e^{-t/2} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + b_2 e^{-t/2} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}
\end{aligned}$$

Summary!

What we did today:

- Direction Fields
- Complex Eigenvalues

Plan for next time:

- Fundamental matrices
- Matrix exponentials
- Repeated eigenvalues