# Math 309 Lecture 6

## The Fundamental Matrix

W.R. Casper

Department of Mathematics
University of Washington

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# Today!

Plan for today:

- Fundamental Matrix
- Matrix-Valued Functions
- Fundamental Matrices for Homogeneous Linear Systems with Constant Coefficients

Next time:

- Repeated Eigenvalues
- Matrix Exponentials
- Fundamental Matrix

# Outline

# Fundamental Matrices

Consider the homogeneous linear system of equations

$$\vec{y}'(t) = A(t)\vec{y}(t)$$

where here $A(t)$ is an $n \times n$ matrix continuous on the interval $(\alpha, \beta)$

- an $n \times n$ matrix $\Psi(t)$ whose column vectors form a fundamental set of solutions on the interval $(\alpha, \beta)$ is called a **fundamental matrix**
- Important note: a fundamental matrix $\Psi(t)$ will be invertible for every $t \in (\alpha, \beta)$ since its column vectors will be linearly independent

# Example

## Question

Find a fundamental matrix for the equation

$$\vec{y}'(t) = A\vec{y}(t), \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

1. the eigenvalues of $A$ are $1, 2$
2. the corresponding eigenspaces are

$$E_1(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \quad E_2(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

# Example

## Question

Find a fundamental matrix for the equation

$$\vec{y}'(t) = A\vec{y}(t), \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

1. this gives us a fundamental set of solutions

$$\begin{pmatrix} e^t \\ 0 \end{pmatrix}, \quad \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

2. therefore we have a fundamental matrix

$$\Psi(t) = \begin{pmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

# Properties of the Fundamental Matrix

Suppose that $\Psi(t)$ is a fundamental matrix for the equation $\vec{y}'(t) = A(t)\vec{y}(t)$ on the interval $(\alpha, \beta)$ where $A(t)$ is continuous. Then the following is true:

(a) $\Psi(t)$ is invertible on the interval $(\alpha, \beta)$

(b) $\Psi(t)$ satisfies the equation $\Psi'(t) = A(t)\Psi(t)$

(c) for all constant vectors $\vec{c}$, $\vec{y}(t) = \Psi(t) \cdot \vec{c}$ is a solution to $\vec{y}'(t) = A(t)\vec{y}(t)$

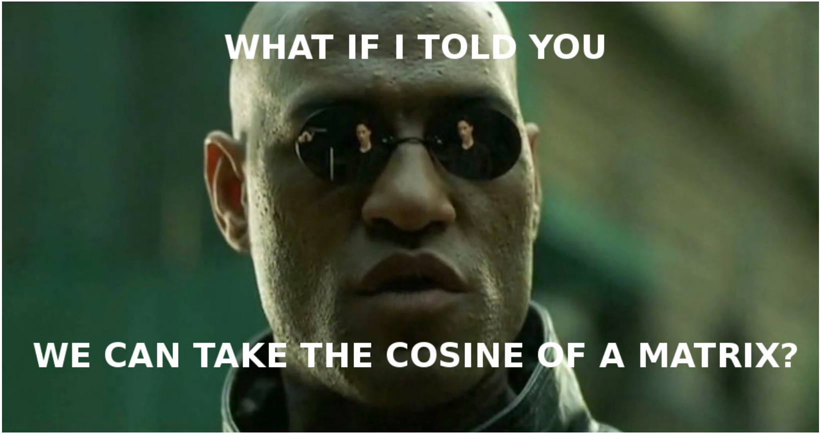
(d) if $t_0 \in (\alpha, \beta)$ and $\vec{v}$ is a constant vector, then $\vec{y}(t) = \Psi(t) \cdot (\Psi(t_0)^{-1}\vec{v})$ is the unique solution of the IVP

$$\vec{y}'(t) = A(t)\vec{y}(t), \quad y(t_0) = \vec{v}.$$

(e) the general solution of $\vec{y}'(t) = A(t)\vec{y}(t)$ is

$$\vec{y}(t) = \Psi(t) \cdot \vec{c}$$

# Morpheus Says



# Matrix Cosine

Consider the Taylor series of $\cos(x)$ based at 0:

$$f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}$$

- we define the **matrix cosine** $\cos(A)$ of an $n \times n$ matrix $A$ by

$$\cos(A) := I - \frac{1}{2}A^2 + \frac{1}{4!}A^4 + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} A^{2j}$$

- let's do an example

# An Example

- Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

- then one may check that

$$A^j = \begin{pmatrix} 1 & 2^j - 1 \\ 0 & 2^j \end{pmatrix}$$

- and therefore

$$\cos(At) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}(At)^{2j} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \begin{pmatrix} t^{2j} & (2t)^{2j} - t^{2j} \\ 0 & (2t)^{2j} \end{pmatrix}$$

# An Example

- Now we know that

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} t^{2j} = \cos(t)$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} (2t)^{2j} = \cos(2t)$$

- therefore the previous expression shows

$$\cos(At) = \begin{pmatrix} \cos(t) & \cos(2t) - \cos(t) \\ 0 & \cos(2t) \end{pmatrix}$$

# Matrix Sine/Matrix Exponential

- in a similar way, we define **matrix sine**

$$\sin(A) := A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \cdots = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j+1)!}A^{2j+1}$$

- and **matrix exponential**

$$\exp(A) := I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!}A^j.$$

- note that Euler's definition still holds for matrices:

$$\exp(iA) = \cos(A) + i\sin(A)$$

# Taylor Series

Given a function $f(x)$ with a Taylor series based at 0

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \cdots = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}x^j$$

for a “sufficiently nice" $n \times n$ matrix $A$, we define

$$f(A) := f(0)I + f'(0)A + \frac{1}{2}f''(0)A^2 + \frac{1}{6}f'''(0)A^3 + \cdots = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}A^j$$

Sufficiently nice means eigenvalues of matrix live within radius of convergence of Taylor series

# Calculation by Hand?

**Question**

For some function $f(x)$ can we calculate $f(A)$ by hand?

- if $D$ is a diagonal matrix, then it's easy!

**Theorem**

If $D$ is a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$, then $f(D)$ is a diagonal matrix with entries $f(d_1), f(d_2), \ldots, f(d_n)$.

- for example

$$A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \implies \cos(A) = \begin{pmatrix} \cos(d_1) & 0 \\ 0 & \cos(d_2) \end{pmatrix}$$

# Diagonalizable Matrices

- Recall that a matrix $A$ is diagonalizable if there exists an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.
- in this case we can also easily calculate $f(A)$!
- observe that $A = PDP^{-1}$ and therefore

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$A^3 = A^2A = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$$

- more generally $A^j = PD^jP^{-1}$

# Diagonalizable Matrices

- from this we see

$$f(A) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} A^j = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} P D^j P^{-1}$$

$$= P \left( \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} D^j \right) P^{-1} = P f(D) P^{-1}.$$

- this gives us the following:

**Theorem**

Suppose that $A$ is diagonalizable with $P^{-1}AP = D$ for some diagonal matrix $D$ and invertible matrix $P$. Then

$$f(A) = Pf(D)P^{-1}.$$

# Example

## Question

Calculate $e^{At}$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

- We know that $A$ has eigenvalues 1 and 2, and eigenspaces

$$E_1(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \quad E_2(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

- therefore we have that

$$P^{-1}AP = D, \text{ for } P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

# Example

- note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

- therefore by our Theorem,

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \exp\begin{pmatrix} t & 0 \\ 0 & 2t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} - e^t \\ 0 & e^{2t} \end{pmatrix} \end{aligned}$$

# Matrix Exponential Properties

- we calcuate the derivative of $e^{At}$:

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\sum_{j=0}^{\infty}\frac{1}{j!}A^j t^j = \sum_{j=0}^{\infty}\frac{1}{j!}jA^j t^{j-1}$$

$$= \sum_{j=1}^{\infty}\frac{1}{(j-1)!}A^j t^{j-1} = \sum_{j=0}^{\infty}\frac{1}{j!}A^{j+1}t^j$$

$$= A\sum_{j=0}^{\infty}\frac{1}{j!}A^j t^j = A\exp(At)$$

- therefore $(e^{At})' = Ae^{At}$

# Matrix Exponential Properties

- Note also that if $B$ is another matrix satisfying $AB = BA$, then

$$e^A e^B = \left(\sum_{j=0}^{\infty} \frac{1}{j!} A^j\right)\left(\sum_{k=0}^{\infty} \frac{1}{k!} B^k\right)$$
$$= \sum_{j,k=0}^{\infty} \frac{1}{(j!)(k!)} A^j B^k = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{1}{(j!)((m-j)!)} A^j B^{m-j}$$
$$= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} \frac{1}{m!} A^j B^{m-j} = \sum_{m=0}^{\infty} \frac{1}{m!} (A+B)^m = e^{A+B}$$

- in particular $(e^A)^{-1} = e^{-A}$

# Fundamental Matrix

- putting this all together, we have that $\Psi(t) = \exp(At)$ satisfies $\Psi'(t) = A\Psi(t)$
- and also that $\Psi(t)$ is nonsinguar, since it has inverse $\exp(-At)$
- therefore the column vectors of $\Psi(t)$ form $n$ linearly independent solutions to $\vec{y}'(t) = A\vec{y}(t)$

**Theorem**

A fundamental matrix of the system $\vec{y}'(t) = A\vec{y}(t)$ on the interval $(-\infty, \infty)$ is $\Psi(t) = \exp(At)$

# Practice

Find the fundamental matrix of the system $\vec{y}'(t) = A\vec{y}(t)$ for each of the following values of $A$

1. $$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$
2. $$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$
3. $$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

# Summary!

What we did today:

- Fundamental Matrices
- Matrix-Valued Functions
- Fundamental Matrices of Homogeneous First-order systems with Constant Coefficients

Plan for next time:

- Nonhomogeneous equations