Math 309 Lecture 7

Diagonalization

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Today!

Plan for today:

- Diagonalization
- Jordan Normal Form
- Calculating a Fundamental Matrix

Next time:

- Nonhomogeneous Differential Equations

Outline

Diagonalizable Matrices

- two matrices A and B are **similar** if there exists an invertible matrix P satisfying $P^{-1}AP = B$
- natural concept – related to change of basis
- a matrix is said to be **diagonalizable** if it is similar to a diagonal matrix
- a matrix which is not diagonalizable is called **defective**

Question

What matrices are diagonalizable?

Eigenbasis

- we have the following theorem:

Theorem

Let A be a diagonal matrix. Then the following are equivalent:

(a) A is diagonalizable

(b) $\mathbb{C}^n$ as a basis consisting of eigenvectors of A (an **eigenbasis**)

(c) for every eigenvalue λ of A, the algebraic and geometric multiplicity of λ are the same

- in other words, A needs “enough" eigenvectors

Special cases

- there are a couple theorems that help us to decide right away if matrices are diagonalizable

Theorem

If all of the eigenvalues of A have algebraic multiplicity 1, then A is diagonalizable

Theorem (Spectral Theorem)

If A is **normal** (ie. A and $A^\dagger$ commute), then A is diagonalizable

- in particular, **Hermitian** ($A = A^\dagger$) and **unitary** ($A^\dagger = A^{-1}$) matrices are diagonalizable

Example

- the following matrix is diagonalizable (why?):

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & -1 \end{pmatrix}$$

- the following matrix is diagonalizable (why?):

$$\begin{pmatrix} 3 & 4 & 9 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$

- the following matrix is NOT diagonalizable (why?):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Use the Eigenbasis

Question

How do we diagonalize a matrix?

- suppose that A is diagonalizable
- let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be an eigenbasis for $\mathbb{R}^n$
- let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the corresponding eigenvalues (resp)
- then $P^{-1}AP = D$ for

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- order is important!!

Example

Question

Find P invertible and D diagonal so that $P^{-1}AP = D$ for

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & -1 \end{pmatrix}$$

Steps:

1. Calculate the eigenvalues (diagonal values of matrix D)
2. For each eigenvalue, find a basis for the eigenspace
3. Put all the bases together to get an eigenbasis for $\mathbb{R}^3$
4. Use them as column vectors in matrix P

Example

1. characteristic poly:

$$p_A(x) = \det(A - xI) = -(x+3)(x-2)(x-8)$$

   Therefore eigenvalues are $-3, 2, 8$

2. corresponding eigenspaces:

$$E_{-3} = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$E_2 = \text{span}\left\{ \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$E_8 = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Example

3. eigenbasis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

4. consequently we have

$$P = \begin{pmatrix} 0 & -5 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Basic Definition

- a **Jordan block** of size m is an $m \times m$ matrix of the form

$$J_m(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

- ie. it is a matrix with some constant value λ on the main diagonal and 1 on the first superdiagonal

Examples

- some examples of **Jordan blocks** include

$$J_1(13) = (13)$$

$$J_2(-7) = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix}$$

$$J_3(-\sqrt{5}) = \begin{pmatrix} \sqrt{5} & 1 & 0 \\ 0 & \sqrt{5} & 1 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$$

$$J_4(\pi) = \begin{pmatrix} \pi & 1 & 0 & 0 \\ 0 & \pi & 1 & 0 \\ 0 & 0 & \pi & 1 \\ 0 & 0 & 0 & \pi \end{pmatrix}$$

Jordan Normal Form

- A matrix B is in **Jordan normal form** if it is in the form

$$B = \begin{pmatrix} J_{m_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{m_2}(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{m_\ell}(\lambda_\ell) \end{pmatrix}$$

- for example, a diagonal matrix is a matrix in Jordan normal form
- other examples include

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Jordan Decomposition

- not every matrix is diagonalizable
- however, *every* matrix is similar to a matrix in Jordan normal form
- the Jordan normal form of a matrix is unique (up to permutation of the Jordan blocks)
- one way to think about this is in terms of **generalized eigenvectors**
- a **generalized eigenvector of rank** k with eigenvalue λ is a nonzero vector in the kernel of $(A - \lambda I)^k$ but not in the kernel of $(A - \lambda I)^{k-1}$
- the dimension of the space of generalized eigenvectors of an eigenvalue is always the same as the algebraic multiplicity
- this gives rise to Jordan normal form

How to Decompose

Question

How do we find P so that $P^{-1}AP$ is in Jordan normal form?

- iteratively!
- for each eigenvalue λ, find a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ of the eigenspace $E_\lambda(A)$
- for each basis vector $\vec{v}_i$, find (if it exists) a vector $\vec{v}_i^{(1)}$ satisfying
$$(A - \lambda I)\vec{v}_i^{(1)} = \vec{v}_i$$
- for each $\vec{v}_i^{(1)}$ we found, find (if it exists) a vector $\vec{v}_i^{(2)}$ satisfying
$$(A - \lambda I)\vec{v}_i^{(2)} = \vec{v}_i^{(1)}$$

How to Decompose

Question

How do we find P so that $P^{-1}AP$ is in Jordan normal form?

- for each $\vec{v}_i^{(2)}$ we found, find (if it exists) a vector $\vec{v}_i^{(3)}$ satisfying

$$(A - \lambda I)\vec{v}_i^{(3)} = \vec{v}_i^{(2)}$$

- ... eventually we'll stop finding these
- the collection $\{\vec{v}_1, \vec{v}_1^{(1)}, \dots, \vec{v}_2, \vec{v}_2^{(2)}, \dots\}$ will be linearly independent with the same size as the algebraic mult. of λ
- use them as columns of P gives us what we want!

Example 1

Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

- char. poly is $(x-1)^2$, so eigenvalues are $1, 1$
- eigenspace:

$$E_1(A) = \text{span}\{\vec{v}\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

- now we find a solution $\vec{v}^{(1)}$ to

$$(A-I)\vec{v}^{(1)} = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 1

Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

- a solution is

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- then take

$$P = \begin{pmatrix} \vec{v} & \vec{v}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

- then

$$P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = J_2(1)$$

Example 2

Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- char. poly is $(x-6)^2$, so eigenvalues are $6, 6$
- eigenspace:

$$E_6(A) = \text{span}\{\vec{v}\} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

- now we find a solution $\vec{v}^{(1)}$ to

$$(A-6I)\vec{v}^{(1)} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\vec{v}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example 2

Question

Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- a solution is

$$\vec{v}^{(1)} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

- then take

$$P = \begin{pmatrix} \vec{v} & \vec{v}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ -1 & 1/2 \end{pmatrix}$$

- then

$$P^{-1}AP = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix} = J_2(6)$$

Review

- recall that a fundamental matrix of

$$\vec{y}'(t) = A\vec{y}(t)$$

- is given by

$$\Psi(t) = \exp(At)$$

- so calculating a fundamental matrix is only as hard as calculating a matrix exponential
- we calculate matrix exponentials by diagonalizing!

Example 1

Question

Find a fundamental matrix for the equation

$$y' = Ay, \quad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- we diagonalize: $P^{-1}AP = D$ for

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Psi(t) = \exp(At) = P\exp(Dt)P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+e^{2t} & 1-e^{2t} \\ 1-e^{2t} & 1+e^{2t} \end{pmatrix}$$

Example 2

Question

Find a fundamental matrix for the equation

$$y' = Ay, \quad A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- A is not diagonalizable!! (oh no)
- how do we calculate e^{At}?
- we can put it into Jordan normal form
- how do we calculate matrix exponential of a Jordan block?

Exponentials of Jordan Blocks

Question

Waht is $\exp(J_m(\lambda)t)$?

$$\exp(J_2(\lambda)t) = \exp(It)\exp(j_2(\lambda)t - It) = \exp(It)\exp\left(\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right)$$

$$= \exp(It)\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

- exponentials of larger Jordan blocks work similarly

Example 2

Question

Find a fundamental matrix for the equation

$$y' = Ay, \quad A = \begin{pmatrix} 7 & 1 \\ -1 & 5 \end{pmatrix}$$

- take

$$P = \begin{pmatrix} 1 & 1/2 \\ -1 & 1/2 \end{pmatrix} \quad D = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix}$$

- therefore

$$\Psi(t) = \exp(At) = P\exp(J_2(6)t)P^{-1}$$

- we can calculate this now...

summary!

what we did today:

- diagonalizable matrices
- jordan normal form
- calculating a fundamental matrix

plan for next time:

- nonhomogeneous differential equations