Math 309 Quiz 1

October 23, 2015

Problem 1. Write what it means for a collection of m vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ to be linearly independent.

Solution 1. A collection of m vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent if the only linear combination equal to the zero vector is the trivial linear combination. More precisely, $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent if the only set of constants $c_1, c_2, \dots, c_m$ satisfying

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$$

is $c_1 = 0, c_2 = 0, c_3 = 0, \dots, c_m = 0$.

Problem 2. Let A be the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}$$

Determine the eigenvalues of A, and for each eigenvalue determine the corresponding eigenspace.

Solution 2. The characteristic polynomial of A is $p_A(x) = x^2 - x - 11$. Therefore the eigenvalues are $(1/2) \pm (3/2)\sqrt{5}$. This means that each eigenvalue has algebraic multiplicity 1, and concequently forces the geometric multiplicity to also be 1. So the eigenspaces must be one dimensional. What are the eigenspaces? We calculate

$$E_{(1/2)+(3/2)\sqrt{5}}(A) = \text{span}\left\{\begin{pmatrix} (1/2) + (1/2)\sqrt{5} \\ 1 \end{pmatrix}\right\}$$

$$E_{(1/2)-(3/2)\sqrt{5}}(A) = \text{span}\left\{\begin{pmatrix} (1/2) - (1/2)\sqrt{5} \\ 1 \end{pmatrix}\right\}$$

Problem 3. Find a fundamental set of solutions for the system

$$x' = 2x + 3y$$
$$y' = 3x - y$$

Solution 3. In terms of vectors, this equation is

$$\vec{y}'(t) = A\vec{y}(t), \quad A = \begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}.$$

Using our data from the previous problem, we have the general solution

$$\vec{y}(t) = c_1 \begin{pmatrix} (1/2) + (1/2)\sqrt{5} \\ 1 \end{pmatrix} e^{((1/2)+(3/2)\sqrt{5})t} + c_2 \begin{pmatrix} (1/2) - (1/2)\sqrt{5} \\ 1 \end{pmatrix} e^{((1/2)-(3/2)\sqrt{5})t}.$$

Problem 4. Write the meaning of the phrase "superposition principle" in the context of systems of homogeneous first order linear ordinary differential equations.

Solution 4. The superposition principle says that if $\vec{w}(t)$ and $\vec{z}(t)$ are solutions to the homogeneous system $\vec{y}'(t) = A(t)\vec{y}(t)$, then any linear combination $c_1\vec{w}(t) + c_2\vec{z}(t)$ is also a solution.