Math 309 Quiz 2

November 24, 2015

Problem 1. Write down the definition of the dimension of a vector space.

Solution 1. The dimension of a vector space is defined to be the number of elements in a basis of the vector space. (This makes sense because all bases for a vector space have the same number of elements).

Problem 2. Write down the definition of a generalized eigenvector.

Solution 2. A generalized eigenvector with eigenvalue λ for a matrix A is a **nonzero** vector $\vec{v}$ satisfying $(A - \lambda I)^m \vec{v} = \vec{0}$ for some positive integer m.

Problem 3. Find a matrix P and a matrix J with P invertible and J in Jordan normal form satisfying $P^{-1}AP = J$, for A the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}.$$

Solution 3. We first calculate the characteristic polynomial

$$p_A(x) = \det(A - xI) = \det \begin{pmatrix} -x & -1 \\ 1 & -2-x \end{pmatrix} = -x(-2-x)+1 = x^2+2x+1 = (x+1)^2.$$

Therefore A has one eigenvalue -1, with algebraic multiplicity 2. The corresponding eigenspace is

$$\begin{aligned} E_1(A) = \mathcal{N}(A - I) &= \mathcal{N}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \\ &= \mathcal{N}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

In particular, the geometric multiplicity of eigenvalue -1 is only one, so A will not be diagonalizable. This already tells us about the Jordan form of A:

$$J = J_2(-1) := \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

To make the matrix P, we need to find a "generalized eigenvector" of A with eigenvalue 1. To do so, we can look for a solution of the equation

$$(A+I)\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This equation has infinitely many solutions – we need only pick one. One solution is $\vec{v} = \binom{1}{0}$. Thus letting

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

then we have $P^{-1}AP = J$.

Problem 4. Calculate e^{At} for the matrix A of the previous problem.

Solution 4. Using the data from the last problem, we have that $e^{At} = Pe^{Jt}P^{-1}$. Then since

$$e^{Jt} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix},$$

we find that

$$e^{At} = Pe^{Jt}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{pmatrix}.$$

ALTERNATIVE SOLUTION: Since $(A+I)^2 = 0$, we have that

$$\exp((A+I)t) = I + (A+I)t.$$

Moreover, since $A+I$ and I commute, we have that

$$\exp(At) = \exp((A+I)t - It) = \exp((A+I)t)\exp(-It) = (I + (A+I)t)e^{-t}.$$

Plugging in the value of A, we obtain

$$\exp(At) = \begin{pmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{pmatrix}.$$